The Scalar Product (Dot Product)

The scalar product (also called 'dot product') $\mathbf{a}\bullet\mathbf{b}$ of two vectors **a** and **b** is a form of vector multiplication. The answer $\mathbf{a}\bullet\mathbf{b}$ is a scalar number (not a vector).

There are two methods of calculating the scalar product. One method is numerical.

Formulae	$\mathbf{a}\bullet\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$, where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Example 1

If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + \mathbf{k}$, calculate $\mathbf{a}\bullet\mathbf{b}$.

Solution

Written as column vectors, $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$.

$$\begin{aligned}\mathbf{a}\bullet\mathbf{b} &= 2\times(-4) + (-3)\times 0 + 5\times 1 \\ &= -8 + 0 + 5 \\ &= -3\end{aligned}$$

The second method for calculating the scalar product is geometric, from a diagram.

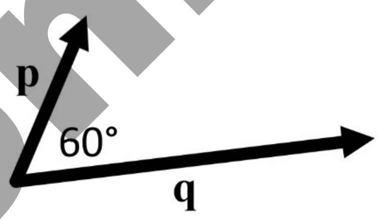
Formula: $\mathbf{a}\bullet\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$, where θ is the angle between **a** and **b**.

Example 2

The diagram shows vectors p and q.

$\lvert\mathbf{p}\rvert = 7$ and $\lvert\mathbf{q}\rvert = 12$.

Calculate $\mathbf{p}\bullet\mathbf{q}$.



Solution

Using the formula, with **p** and **q**, we get:

$$\begin{aligned}\mathbf{p}\bullet\mathbf{q} &= \lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\cos\theta \\ &= 7\times 12\times\cos 60^\circ \\ &= 84 \times\frac{1}{2} \\ &= 42\end{aligned}$$

SAMPLE for evaluation only

Vectors and Angles

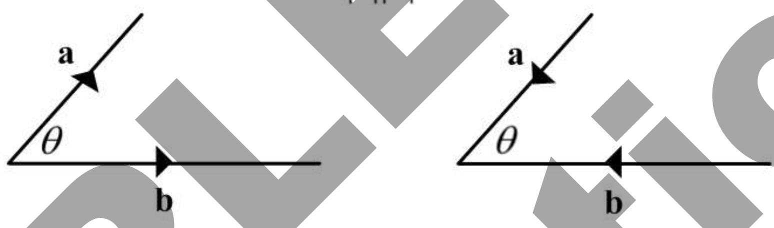
The scalar product is linked to the angle between two vectors by the following formula:

Formula: $\mathbf{a}\bullet\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**.

This can be rearranged to give $\cos\theta=\dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ or alternatively $\theta=\cos^{-1}\left(\dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$

This formula is given on the formula sheet. It assumes that the vectors **a** and **b** are positioned so that they either both point away from the angle, or both point into the angle.

$\mathbf{a}\bullet\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$



Both vectors pointing away from the angle

Both vectors pointing into the angle

Formula (not given on the formula sheet)
If one vector is pointing away from the angle, whilst the other points into the angle, then the formula changes to be $\mathbf{a}\bullet\mathbf{b}=-|\mathbf{a}||\mathbf{b}|\cos\theta$.

$\mathbf{a}\bullet\mathbf{b}=-|\mathbf{a}||\mathbf{b}|\cos\theta$

One vector pointing into the angle.
The other vector pointing away from the angle.

<u>Example 1 – finding the angle between two vectors</u>

Calculate the angle between the vectors $\underline{\mathbf{a}}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$ **and** $\underline{\mathbf{b}}=\begin{pmatrix}3\\2\\-1\end{pmatrix}$**.**

Solution

First we calculate $|\mathbf{a}|$, $|\mathbf{b}|$ and $\mathbf{a}\bullet\mathbf{b}$.

$$|\mathbf{a}|=\sqrt{2^2+3^2+(-1)^2}=\sqrt{14}$$

$$|\mathbf{b}|=\sqrt{3^2+2^2+(-1)^2}=\sqrt{14}$$

$$\begin{aligned}\mathbf{a}\bullet\mathbf{b}&=2\times3+3\times2+(-1)\times(-1)\\&=6+6+1\\&=13\end{aligned}$$

Then use the formula $\cos\theta=\dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

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SAMPLE for evaluation only

(Example 1 continued)

$$\cos\theta = \frac{13}{\sqrt{14}\sqrt{14}}$$

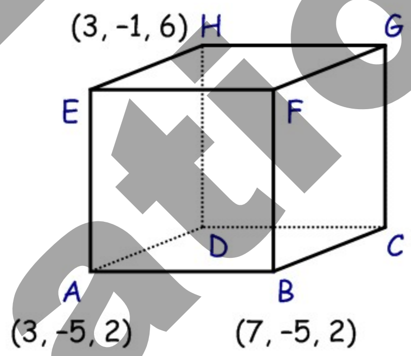
$$\cos\theta = \frac{13}{14}$$

$$\theta = \cos^{-1}\left(\frac{13}{14}\right)$$

$$= \underline{\underline{21\cdot 8^\circ}} \text{ (1 d.p.)}$$

Example 2 – finding the angle from coordinates

ABCDEFGH is a cube with sides of length 4 units.
A is the point (3, –5, 2), B is the point (7, –5, 2) and H is the point (3, –1, 6).



(a) Express $\overrightarrow{BA}$ and $\overrightarrow{BH}$ in component form.
(b) Hence calculate the size of angle ABH.

Solution

(a) We subtract the position vectors to convert coordinates into vectors.

$$\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}$$

$$\overrightarrow{BH} = \mathbf{h} - \mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix}$$

(b) First calculate $\left|\overrightarrow{BA}\right|$, $\left|\overrightarrow{BH}\right|$ and $\overrightarrow{BA} \bullet \overrightarrow{BH}$:

$$\left|\overrightarrow{BA}\right| = \sqrt{(-4)^2 + 0^2 + 0^2} = 4$$

$$\left|\overrightarrow{BH}\right| = \sqrt{(-4)^2 + 4^2 + 4^2} = \sqrt{48}$$

$$\overrightarrow{BA} \bullet \overrightarrow{BH} = (-4)\times(-4) + 0\times 4 + 0 \times 4 = 16$$

Then $$\cos\theta = \frac{\overrightarrow{BA} \bullet \overrightarrow{BH}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BH}\right|}$$

$$\cos\theta = \frac{16}{4\sqrt{48}}$$

$$\theta = \cos^{-1}\left(\frac{16}{4\sqrt{48}}\right)$$

$$= \underline{54\cdot 7^\circ} \text{ (1 d.p.)}$$

SAMPLE for evaluation only