Example 3 – values of *k*

A circle has equation $x^2+y^2+4x+6y+k=0$. Find the range of possible values of *k*.

Solution

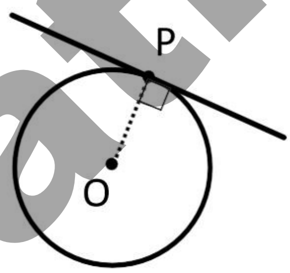
In this equation, $g = 2$, $f = 3$ and $c = k$.

The circle exists if and only if $g^2+f^2-c>0$. Therefore:

$$\begin{aligned} g^2+f^2-c&>0\\ 2^2+3^2-k&>0\\ 13-k&>0\\ 13&>k\\ k&<13 \end{aligned}$$

Equation of a Tangent to a Circle

A tangent to a circle is a straight line that just 'touches' the edge of the circle, as shown in the diagram. For National 5 Maths, you learnt that a tangent to a circle is perpendicular to the radius.



A tangent is a straight line. Therefore to find its equation we use the usual formula $y-b=m(x-a)$ and the facts about perpendicular gradients on page 126.

Fact (from National 5 Maths)
The tangent to a circle (centre O) at point P is perpendicular to the radius OP.

Therefore $m_{\text{tangent}} \times m_{\text{radius}} = -1$

Example

(a) Show that the point P(5, –1) lies on the circle with equation $(x-6)^2+(y-1)^2=5$.
(b) Find the equation of the tangent to the circle at point P.

Solution

(a) We substitute $x = 5$ and $y = -1$ into the left-hand side of the circle equation:

$$\begin{aligned} \text{LHS} &= (x-6)^2+(y-1)^2\\ &= (5-6)^2+(-1-1)^2\\ &= (-1)^2+(-2)^2\\ &= 5 = \text{RHS} \end{aligned}$$

LHS = RHS, therefore the point satisfies the equation, therefore P lies on the circle.

(b) Step one: identify the coordinates of a point on the tangent

The point P lies on the tangent, so we use (5, –1).

Step two: calculate the gradient of the radius: $m_{\text{radius}} = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{1-(-1)}{6-5} = \dfrac{2}{1} = 2$

(continued on next page)

(Example continued)

<u>Step three:</u> use the fact that the radius and tangent are perpendicular to write down the gradient of the tangent.

$$m_{radius} \times m_{tangent} = -1.$$

Therefore $m_{\text{tangent}} = -\frac{1}{2}$.

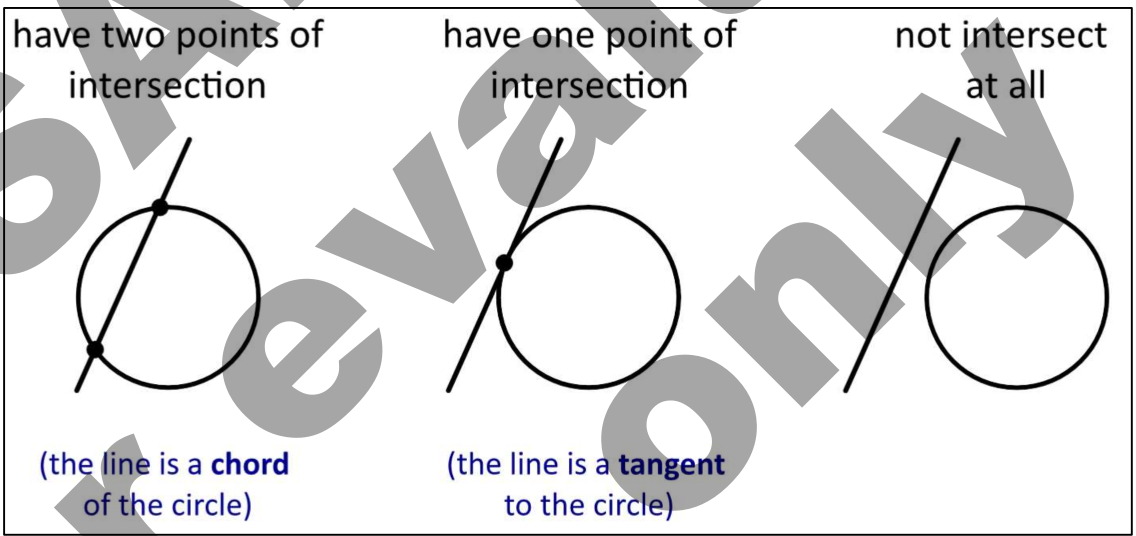
<u>Step four:</u> use the equation $y - b = m(x - a)$ to find the equation of the tangent.

We use the point (5, –1) and the gradient $-\frac{1}{2}$:

$$\begin{aligned} y - b &= m(x-a) \\ y-(-1) &= -\frac{1}{2}(x-5) \\ 2(y+1) &= -(x-5) \\ 2y+2 &= -x+5 \\ x+2y-3 &= 0 \end{aligned}$$

Intersection of a Circle and a Straight Line

There are three possible situations that may arise with a circle and a straight line as shown in the diagram below:



When finding the intersection of a line and a circle you will eventually end up with a quadratic equation that needs to be solved (the 'equation of intersection'). The discriminant of the equation of intersection tells us which of these three cases applies:

- **Discriminant is positive:** line and circle have two points of intersection.
- **Discriminant is zero:** line is tangent to circle.
- **Discriminant is negative:** line and circle do not intersect.

We find the points of intersection by substituting the equation of the line into the equation of the circle. The equation of the line must be rearranged to have either *x* or *y* as the subject.