Vector Pathways

We can use the rules of adding and taking away vectors to express a vector $\overrightarrow{AB}$ in a geometrical situation as a combination of other, known, vectors.

To do this, we identify a route, or pathway, between A and B, in which each step of the route can be expressed in terms of one of the other known pathways.

- Where a vector is followed 'backwards', it is subtracted.
- Where a vector is followed 'forwards', it is added.

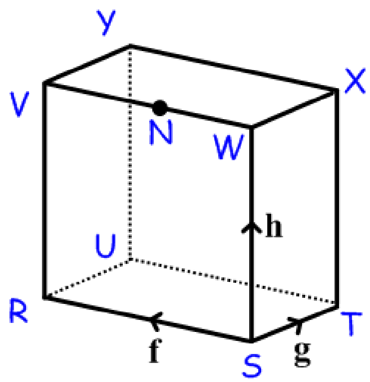
We can choose any valid route, and the final answer, when simplified, will always be the same.

Example 1 (adapted from 2008 SQA exam question)

In the diagram, RSTUVWXY represents a cuboid. $\overrightarrow{SR}$ represents vector $\underline{f}$, $\overrightarrow{ST}$ represents vector $\underline{g}$ and $\overrightarrow{SW}$ represents vector $\underline{h}$.

N is the midpoint of VW.

Express $\overrightarrow{RX}$ and $\overrightarrow{TN}$ in terms of $\underline{f}$, $\underline{g}$ and $\underline{h}$.



Solution

For $\overrightarrow{RX}$:

Step one: identify a pathway from R to X.

One possible pathway is $\overrightarrow{RU}, \overrightarrow{UY}, \overrightarrow{YX}$.

Step two: express each part of the pathway in terms of a known vector

$\overrightarrow{RU} = \mathbf{g}$, $\overrightarrow{UY} = \mathbf{h}$, $\overrightarrow{YX} =$ backwards along $\mathbf{f}$

Therefore $\overrightarrow{RX} = \mathbf{g} + \mathbf{h} - \mathbf{f}$ (or $-\mathbf{f} + \mathbf{g} + \mathbf{h}$ or any other algebraically equivalent expression).

For $\overrightarrow{TN}$:

Step one: identify a pathway from T to N:

One possible pathway is $\overrightarrow{TX}, \overrightarrow{XW}, \overrightarrow{WN}$.

Step two: express each part of the pathway in terms of a known vector:

$\overrightarrow{TX} = \mathbf{h}$, $\overrightarrow{XW} =$ backwards along $\mathbf{g}$, $\overrightarrow{WN} =$ half way along $\mathbf{f}$

Therefore $\overrightarrow{TN} = \mathbf{h} - \mathbf{g} + \frac{1}{2}\mathbf{f}$ (or $\frac{1}{2}\mathbf{f} - \mathbf{g} + \mathbf{h}$ or any other algebraically equivalent expression).

Example 2 – from component form

Three vectors can be expressed as follows:

- $\overrightarrow{PQ} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
- $\overrightarrow{QR} = -4\mathbf{i} + \mathbf{j} + 7\mathbf{k}$
- $\overrightarrow{MR} = 2\mathbf{i} - \mathbf{k}$

(a) Find $\overrightarrow{PR}$.

(b) Hence of otherwise, find $\overrightarrow{PM}$.

Solution

(a) The question is asking us to find the vector from P to R.
We must find a route from P to R.
A possible route is 'forwards from P to Q, and then forwards from Q to R'.

This can be expressed as $\overrightarrow{PQ} + \overrightarrow{QR}$:

$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PQ} + \overrightarrow{QR} \\ &= (3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) + (-4\mathbf{i} + \mathbf{j} + 7\mathbf{k}) \\ &= -\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}\end{aligned}$$

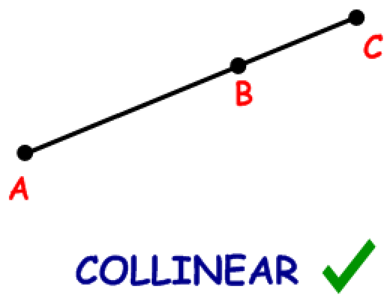
(b) The question is asking us to find the vector from P to M.
We must find a route from P to M.
A possible route is 'forwards from P to R, and then forwards from R to M'.
This is also expressed as 'forwards from P to R, and then **backwards** from M to R'.

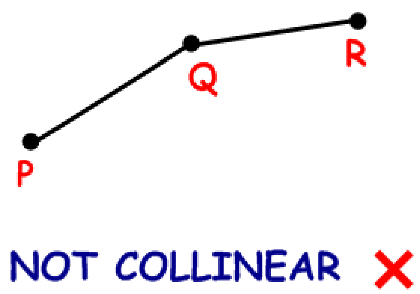
This can be expressed as $\overrightarrow{PR} - \overrightarrow{MR}$:

$$\begin{aligned}\overrightarrow{PM} &= \overrightarrow{PR} - \overrightarrow{MR} \\ &= (-\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}) - (2\mathbf{i} - \mathbf{k}) \\ &= -3\mathbf{i} - 4\mathbf{j} + 10\mathbf{k}\end{aligned}$$

Collinear Points and Ratio

Three or more points are said to be **collinear** if they lie on the same straight line:





In three-dimensions, if three points A, B and C are collinear, then the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are all parallel (recall that two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = k\mathbf{v}$ for some $k \neq 0$).

To show whether three points P, Q and R are collinear, we must therefore show that:

- $\overrightarrow{PQ} = k\overrightarrow{QR}$ for some $k \neq 0$ (or $\overrightarrow{PQ} = k\overrightarrow{PR}$, or $\overrightarrow{QR} = k\overrightarrow{PR}$: any pair of vectors is fine, so long as they contain a common point).
- It is crucial that the two vectors have a common point (in this case Q), otherwise it would just mean that they were parallel, not that they are collinear.

For a sufficient conclusion at the end of a collinearity proof, it is essential to use the words **'parallel'** and **'common point'**.

Example 1 – three dimensions

Prove that A(3, 4, 1), B(9, 1, –5) and C(11, 0, –7) are collinear.

Solution

First find $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix}$$

$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} 11 \\ 0 \\ -7 \end{pmatrix} - \begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

$\overrightarrow{AB} = 3\overrightarrow{BC}$, so $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

B is a common point, so A, B and C are collinear. ■

In two dimensions:

- If points A, B and C are collinear, then AB, AC and BC all have the same **gradient**.
- If points P, Q and R are not collinear, then PQ, QR and PR will all have different gradients.

To show whether three points A, B and C are collinear, we must therefore show that AB and BC (or AB & AC or AC & BC: any pair of points is fine so long as the two line segments chosen have a common point) have the same gradient. If they have the same gradient, and since B lies on both line segments, we can conclude that the lines are collinear.

Example 2 – two dimensions

Show that the points P(–6, –1), Q(0, 2) and R(8, 6) are collinear.

Solution

$$m_{PQ} = \frac{2-(-1)}{0-(-6)} = \frac{3}{6} = \frac{1}{2}$$

$$m_{QR} = \frac{6-2}{8-0} = \frac{4}{8} = \frac{1}{2}$$

The two line segments are parallel and Q is a common point, therefore P, Q and R are collinear. ■