Methods in Algebra and Calculus Unit

Topic 1·1: Partial Fractions

Algebraic Long Division

Long division is a written division algorithm where written subtraction sums are also used to calculate the remainders at each stage (instead of calculating them mentally). We use long division when we have an **improper** (also known as '**top-heavy**') fraction where the numerator is greater than the denominator, such as $\frac{7}{4}$ or $\frac{23}{5}$.

It is not just numerical fractions that can be improper/top-heavy. There are also **improper algebraic fractions**. An improper algebraic fraction is a fraction in which where the biggest power of *x* in the numerator is greater than or equal to the biggest power of *x* in the denominator.

Examples of improper algebraic fractions.	$\frac{x^2+1}{x}$, $\frac{x^5}{x^4+2x+1}$, $\frac{x^3+2x+1}{x^3-5x+4}$
Examples of algebraic fractions that are not improper.	$\frac{x}{x^2+1}$, $\frac{x^4+2x+1}{x^5}$

When we have an improper fraction, we need to be able to split it up into a quotient (something that is not a fraction), plus a proper fraction. A numerical example might be to split $\frac{23}{5}$ up to be $4+\frac{3}{5}$. To do this with an algebraic fraction, we must use algebraic long division.

In short, the method involves repeatedly following a sequence of steps:

(1) **divide** the <u>first</u> term from the numerator by the <u>first</u> term from the denominator.
(2) **multiply** the result of step (1) by the <u>full</u> denominator.
(3) **subtract** the result of step (2) from the numerator to create a new numerator which will be used in the next step.
(4) Repeat steps (1), (2) and (3) until the highest power in the numerator is less than the highest power in the denominator.
   - When we stop, the final 'new' numerator is the remainder.

<u>Example 1</u>

Express $\frac{3x^2+5x+7}{x+1}$ as the sum of a polynomial function and a proper rational function.

Solution

<u>Initial step:</u> Lay out the division as shown on the right.

$$x+1\overline{)3x^2+5x+7}$$

(continued on the next page)

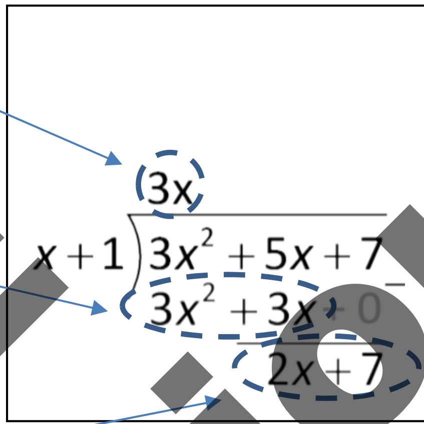
(Example 1 continued)

Iteration 1: we follow steps (1), (2) and (3) as listed above.

Step 1: Divide the first term from the numerator ($3x^2$) by the first term from the denominator (x).

Answer: $3x$.

Write this at the top above the $3x^2$.



Step 2: multiply the result of step 1a ($3x$) by the full denominator ($x+1$).

Answer: $3x^2+3x$.

Write this below the $3x^2+5x$.

Step 3: subtract the result of step 1b ($3x^2+3x$) from the numerator ($3x^2+5x+7$) to create a new numerator which will be used in the next step.

Answer: the new numerator is $2x+7$.

Write this as a subtraction sum below the $3x^2+3x$.

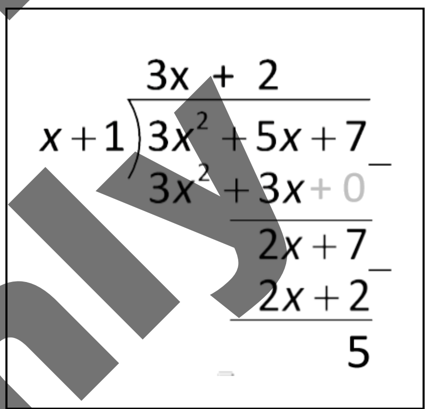
The highest power in the numerator is not yet lower than the highest power in the denominator, so we continue for another iteration.

Iteration 2: we again follow steps (1), (2) and (3) as listed above.

Step 1: Divide the first term from the **new** numerator ($2x$) by the first term from the denominator (x).

Answer: (+)2.

Write this at the very top above the $2x$.



Step 2: multiply the result of step 1a (2) by the full denominator ($x+1$).

Answer: $2x+2$.

Write this below the $2x+7$.

Step 3: subtract the result of step 1b ($2x+2$) from the **new** numerator ($2x+7$) to create a new numerator which will be used in the next step.

Answer: the new numerator is 5.

Write this as a subtraction sum below the $2x+7$.

The highest power in the numerator is now lower than the highest power in the denominator, so we stop.

When writing our answer:

- The quotient (the very top line of the written division) is $3x+2$.
- The remainder (the final numerator) is 5. This goes on top of the fraction part of the answer.
- The denominator of the fraction part of the answer is the denominator of the original question.

Final answer: $\dfrac{3x^2+5x+7}{x+1}=\underline{\underline{3x+2+\dfrac{5}{x+1}}}$.

Example 2

Express $\dfrac{x^3+4x^2-x+2}{x^2+x}$ as the sum of a polynomial and a proper rational function.

Solution

The full working (following the steps outlined in the previous example) is as shown below:

$$
\begin{array}{r|l}
 & \quad x \quad +3 \\
\hline
x^2+x & x^3+4x^2-x+2 \quad - \\
 & x^3+x^2+0+0 \\
\hline
 & \qquad 3x^2-x+2 \quad - \\
 & \qquad 3x^2+3x+0 \\
\hline
 & \qquad\qquad -4x+2
\end{array}
$$

Final answer: $\dfrac{x^3+4x^2-x+2}{x^2+x} = \underline{\underline{x+3+\dfrac{-4x+2}{x^2+x}}}$.

Partial Fractions

You learnt at National 5 to add or subtract algebraic fractions to express them as a single fraction (possibly using a method known as 'kiss and smile') . e.g. $\dfrac{2}{x+3}+\dfrac{3}{x+4}=\dfrac{5x+17}{(x+3)(x+4)}$.
This method takes two (or more) 'simpler' fractions and combines them into one more complicated rational function.

The method of partial fractions goes backwards with this method: it takes a more complicated rational function and breaks it down again to be a sum of two 'simpler' fractions. We need this method in order to differentiate or integrate many functions.

The method is:

- Initial step: If the rational function is top-heavy, use algebraic long division to re-express it.
- Step one: Factorise the denominator, if not already done.
- Step two: Identify which 'type' of factors are in the denominator (see the three types listed in this chapter).
- Step three: Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values *A*, *B*, *C* etc. in the numerators.
- Step four: Multiply both sides by the denominator of the original fraction.
- Step five: Solve equations to obtain values for *A*, *B*, *C* etc. The simplest way to obtain the values is to substitute in various values of *x*, usually the roots of the denominator. However any value of *x* could be substituted.

TYPE ONE: Distinct linear factors: When every factor in the denominator is linear and distinct, each factor forms the denominator of a fraction in the final answer, with a constant on top.

$$\frac{f(x)}{(x+p)(x+q)}=\frac{A}{x+p}+\frac{B}{x+q} \qquad\qquad \frac{g(x)}{(ax+b)(cx+d)(ex+f)}=\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{ex+f}$$

For example: $\dfrac{x+7}{(3x-2)(x+1)} = \dfrac{A}{3x-2} + \dfrac{B}{x+1}$ or $\dfrac{3x+2}{x^2+5x+6} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$.

Example 1

Express $\dfrac{x+7}{x^2-x-2}$ in partial fractions.

Solution

Initial step: If the rational function is top-heavy, use algebraic long division to re-express it.

This fraction is not top-heavy, so this step is not required.

Step one: Factorise the denominator, if not already done.

$$\frac{x+7}{x^2-x-2} = \frac{x+7}{(x-2)(x+1)}$$

Step two: Identify which 'type' of factors are in the denominator.

Both factors are linear and distinct, so this is 'type one'.

Step three: Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values A, B, C etc. in the numerators.

$$\frac{x+7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Step four: Multiply both sides by the denominator of the original fraction and simplify.

$$\frac{(x+7)(x-2)(x+1)}{(x-2)(x+1)} = \frac{A(x-2)(x+1)}{x-2} + \frac{B(x-2)(x+1)}{x+1}$$

$$x+7 = A(x+1) + B(x-2) \quad (**)$$

Step five: Solve equations to obtain values for A, B, C etc. The simplest way to obtain the values is to substitute in various values of x, usually the roots of the denominator. However any value of x could be substituted.

The roots of the two brackets are x = 2 and x = –1.

First substitute x = 2 into the expression marked (**) above:

$$2+7 = A(2+1) + B(2-2)$$
$$9 = A(3) + B(0)$$
$$3A = 9$$
$$A = 3$$

Now substitute x = –1 into the expression marked (**) above:

$$-1+7 = A(-1+1) + B(-1-2)$$
$$6 = A(0) + B(-3)$$
$$-3B = 6$$
$$B = -2$$

Answer: A = 3 and B = –2, so $\dfrac{x+7}{(x-2)(x+1)} = \dfrac{3}{x-2} - \dfrac{2}{x+1}$.

TYPE TWO: Repeated linear factors: When all factors in the denominator are linear but one of the factors appears more than once (i.e. brackets with powers, including a factor of x^n), then the factor that is repeated requires more than one partial fraction as shown below:

$$\frac{f(x)}{x^2(x+p)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+p} \qquad \frac{g(x)}{(ax+b)(cx+d)^2}=\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}$$

For example: $\frac{x^2-7x+9}{(x+2)(3x-1)^2}=\frac{A}{x+2}+\frac{B}{3x-1}+\frac{C}{(3x-1)^2}$ or $\frac{2}{x^2(2x-5)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{2x-5}$.

Example 2

Express $\frac{x^2-7x+9}{(x+2)(x-1)^2}$ in partial fractions.

Solution

Initial step: If the rational function is top-heavy, use algebraic long division.

This fraction is not top-heavy, so this step is not required.

Step one: Factorise the denominator, if not already done.

On this occasion, the denominator is already factorised.

Step two: Identify which 'type' of factors are in the denominator.

Both factors are linear but the factor $(x-1)^2$ is repeated, so this is 'type two'.

Step three: Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values *A*, *B*, *C* etc. in the numerators.

$$\frac{x^2-7x+9}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Step four: Multiply both sides by the denominator of the original fraction and simplify.

$$\frac{(x^2-7x+9)\cancel{(x+2)(x-1)^2}}{\cancel{(x+2)(x-1)^2}}=\frac{A\cancel{(x+2)}(x-1)^2}{\cancel{x+2}}+\frac{B(x+2)(x-1)^{\cancel{2}}}{\cancel{x-1}}+\frac{C(x+2)\cancel{(x-1)^2}}{\cancel{(x-1)^2}}$$

$$x^2-7x+9 \quad =A(x-1)^2+B(x+2)(x-1)+C(x+2) \qquad (**)$$

Step five: Solve equations to obtain values for *A*, *B*, *C* etc. The simplest way to obtain the values is to substitute in various values of *x*, usually the roots of the denominator. However any value of *x* could be substituted.

The roots of the two brackets are x = –2 and x = 1. We will also need a third value of x as there are three constants to find. We could use any value, but it makes sense to choose x = 0 as that is an 'easy' number.

First substitute x = –2 into the expression marked (**) above:

$$x^2-7x+9=A(x-1)^2+B(x+2)(x-1)+C(x+2) \qquad (**)$$
$$(-2)^2-7(-2)+9=A(-2-1)^2+B(-2+2)(-2-1)+C(-2+2)$$
$$4+14+9=A(-3)^2+B(0)(-3)+C(0)$$
$$27=9A$$
$$A=3$$

(continued on next page)

(Example 2 continued)

Now substitute x = 1 into the expression marked (**) above:

$$x^2 - 7x + 9 = A(x-1)^2 + B(x+2)(x-1) + C(x+2) \qquad (**)$$
$$1^2 - 7(1) + 9 = A(1-1)^2 + B(1+2)(1-1) + C(1+2)$$
$$1 - 7 + 9 = A(0)^2 + B(3)(0) + C(3)$$
$$3 = 3C$$
$$C = 1$$

Now substitute x = 0 into the expression marked (**) above, ***remembering that we already know that A = 3 and C = 1****:*

$$x^2 - 7x + 9 = A(x-1)^2 + B(x+2)(x-1) + C(x+2) \qquad (**)$$
$$0^2 - 7(0) + 9 = 3(0-1)^2 + B(0+2)(0-1) + 1(0+2)$$
$$9 = 3(-1)^2 + B(2)(-1) + 1(2)$$
$$9 = 3 - 2B + 2$$
$$9 = 5 - 2B$$
$$2B = 5 - 9$$
$$2B = -4$$
$$B = -2$$

Answer: A = 3, B = –2 and C = 1, so $\dfrac{x^2 - 7x + 9}{(x+2)(x-1)^2} = \dfrac{3}{x+2} - \dfrac{2}{x-1} + \dfrac{1}{(x-1)^2}$.

An **irreducible** factor is one that cannot be factorised. A quadratic factor is irreducible if the discriminant is not a perfect square.

TYPE THREE: Irreducible quadratic factors: When one of the factors in the denominator is a quadratic that cannot be factorised (e.g. $x^2 + 5$ or $x^2 + x + 1$), then the partial fraction corresponding to that factor has a numerator of the form $Ax + B$ (as opposed to just 'A' or 'B').

For example: $\dfrac{3x-1}{(x+5)(x^2+3)} = \dfrac{A}{x+5} + \dfrac{Bx+C}{x^2+3}$ or $\dfrac{x^2-7x+9}{(x^2+2x+8)(x-4)} = \dfrac{Ax+B}{x^2+2x+8} + \dfrac{C}{x-4}$.

Example 3

Express $\dfrac{3x^2 + 2x + 1}{(x+1)(x^2+2x+2)}$ in partial fractions.

Solution

Initial step: If the rational function is top-heavy, use algebraic long division.

This fraction is not top-heavy, so this step is not required.

Step one: Factorise the denominator, if not already done.

On this occasion, the denominator is already factorised.

Step two: Identify which 'type' of factors are in the denominator.

(continued on next page)

(Example 3 continued)

There is a quadratic factor x^2+2x+2. *We double-check that this is irreducible using the discriminant:*

$$b^2-4ac=2^2-4\times1\times2=-4$$

$b^2-4ac<0$, *so* x^2+2x+2 *<u>is</u> irreducible, so this is 'type three'.*

<u>Step three:</u> Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values *A*, *B*, *C* etc. in the numerators.

$$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2}$$

<u>Step four:</u> Multiply both sides by the denominator of the original fraction and simplify.

$$\frac{(3x^2+2x+1)\cancel{(x+1)(x^2+2x+2)}}{\cancel{(x+1)(x^2+2x+2)}}=\frac{A\cancel{(x+1)}(x^2+2x+2)}{\cancel{x+1}}+\frac{(Bx+C)(x+1)\cancel{(x^2+2x+2)}}{\cancel{x^2+2x+2}}$$

$$3x^2+2x+1=A(x^2+2x+2)+(Bx+C)(x+1)\quad(**)$$

<u>Step five:</u> Solve equations to obtain values for *A*, *B*, *C* etc.

The root of the linear bracket is x = –1. The irreducible factor has no roots, so we need two further x-values to substitute. We could use any value, but it makes sense to choose x = 0 and x = 1 as they are 'easy' numbers.

First substitute x = –1 into the expression marked (**) above:

$$3x^2+2x+1=A(x^2+2x+2)+(Bx+C)(x+1)$$
$$3(-1)^2+2(-1)+1=A((-1)^2+2(-1)+2)+(B(-1)+C)(-1+1)$$
$$3-2+1=A(1-2+2)+(-B+C)(0)$$
$$2=A$$

Now substitute x = 0 into the expression marked (**) above ***remembering that we already know that A = 2****:*

$$3x^2+2x+1=A(x^2+2x+2)+(Bx+C)(x+1)$$
$$3(0)^2+2(0)+1=2(0^2+2(0)+2)+(B(0)+C)(0+1)$$
$$1=2(2)+(C)(1)$$
$$1=4+C$$
$$C=-3$$

Now substitute x = 1 into the expression marked (**) above, ***remembering that we already know that A = 2 and C = –3****:*

$$3x^2+2x+1=A(x^2+2x+2)+(Bx+C)(x+1)$$
$$3(1)^2+2(1)+1=2(1^2+2(1)+2)+(B(1)-3)(1+1)$$
$$3+2+1=2(5)+(B-3)(2)$$
$$6=10+2(B-3)$$
$$B=1\quad\text{(some working omitted)}$$

Answer: A = 2, B = 1 and C = –3, so $\dfrac{3x^2+2x+1}{(x+1)(x^2+2x+2)}=\dfrac{2}{x+1}+\dfrac{x-3}{x^2+2x+2}$.

When the fraction is improper, algebraic division (see page 11) must be used first to express the improper fraction as a polynomial and a proper fraction.

Example 4

Express $\frac{x^3-3x}{x^2-x-2}$ as the sum of a polynomial and partial fractions.

Solution

Step A: use algebraic long division:

$$\begin{array}{r|l} & x \quad + 1 \\ \hline x^2-x-2 & x^3+0x^2-3x+0 \\ & x^3-x^2-2x+0 \\ \hline & \qquad x^2-x+0 \\ & \qquad x^2-x-2 \\ \hline & \qquad\qquad 2 \end{array}$$

Hence $\frac{x^3-3x}{x^2-x-2}=x+1+\frac{2}{x^2-x-2}$. (*)

Step B: Express $\frac{2}{x^2-x-2}$ in partial fractions.

Using the standard method from page 14 (this is a 'type one' fraction).

$$\frac{2}{x^2-x-2}=\frac{2}{(x-2)(x+1)}$$

$$\frac{2}{x^2-x-2}=\frac{2}{(x-2)(x+1)}$$

$$\frac{2}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$$

$2=A(x+1)+B(x-2)$ (a step of working ('cancelling') was omitted here)

When $x = 2$:

$2=A(2+1)+B(2-2)$

$2=3A+0$

$A=\frac{2}{3}$

when $x = -1$

$2=A(-1+1)+B(-1-2)$

$2=0-3B$

$B=-\frac{2}{3}$

Hence $\frac{2}{x^2-x-2}=\frac{\frac{2}{3}}{x-2}-\frac{\frac{2}{3}}{x+1}=\frac{2}{3(x-2)}-\frac{2}{3(x+1)}$.

Step C: go back to formula (*) to re-express the full function:

Final answer: $\frac{x^3-3x}{x^2-x-2}=x+1+\frac{2}{3(x-2)}-\frac{2}{3(x+1)}$.