Integrals Where the Solutions Involve Inverse Trigonometric Functions

Formulae (these are given on the formula sheet)

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

Example 1

Integrate $\int \frac{5}{x^2+49}\,dx$.

Solution

$$\int \frac{5}{x^2+49}\,dx = 5\int \frac{1}{x^2+49}\,dx \qquad \text{(taking the factor of 5 outside the integral sign)}$$

$$= 5\int \frac{1}{7^2+x^2}\,dx \qquad \text{(rearranging the denominator)}$$

$$= 5\cdot\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+C$$

$$= \frac{5}{7}\tan^{-1}\left(\frac{x}{7}\right)+C$$

Example 2 – requiring substitution

Integrate $\int \frac{6}{\sqrt{4-9x^2}}\,dx$.

Solution

$$\int \frac{6}{\sqrt{4-9x^2}}\,dx = 6\int \frac{1}{\sqrt{2^2-(3x)^2}}\,dx$$

We use integration by substitution using the substitution $u=3x$.

$u=3x \Rightarrow \frac{du}{dx}=3 \Rightarrow dx=\frac{du}{3}$. Then:

$$\int \frac{6}{\sqrt{2^2-(3x)^2}}\,dx = \int \frac{6}{\sqrt{2^2-u^2}}\cdot\frac{du}{3}$$

$$= \frac{6}{3}\int \frac{1}{\sqrt{2^2-u^2}}\,du$$

$$= 2\cdot\sin^{-1}\left(\frac{u}{2}\right)+C \qquad \text{using } \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

$$= 2\sin^{-1}\left(\frac{3x}{2}\right)+C \qquad \text{resubstituting } u=3x$$