The Discriminant and Quadratic Equations

Formula

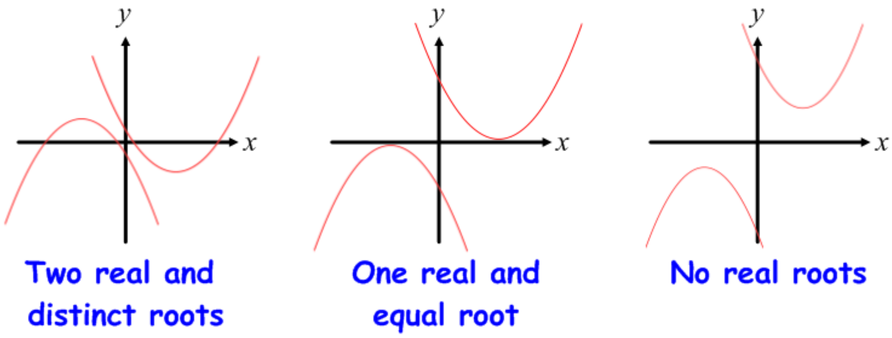
The solutions to the quadratic equation $ax^2 + bx + c$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

You learnt at National 5 that the **discriminant** of the quadratic equation $ax^2 + bx + c$ is the number that has to be square rooted to complete the quadratic formula. The symbol Δ can be used for the discriminant.

Formula

The discriminant of a quadratic equation $ax^2 + bx + c$ is given by the formula $\Delta = b^2 - 4ac$

A quadratic equation may have two distinct real roots, one equal real root or no real roots.



Simply put, the discriminant tells us how many real roots there are to a quadratic equation:

- If the discriminant is positive ($\Delta > 0$), the equation has two **distinct, real** roots (also known as **unequal real** roots).
- If the discriminant is zero ($\Delta = 0$), the equation has one **real and equal** (or **real and repeated**) root.
- If the discriminant is negative ($\Delta < 0$), the equation has **no real roots**.

Note: for the roots to be just **real** (i.e. no distinction between whether the roots are distinct or equal), then the discriminant may be positive or zero ($\Delta \geq 0$) – so long as it isn't negative, the roots are real.

The discriminant can also tell us whether roots are irrational (a non-repeating decimal) or rational (a whole number or fraction):

- If the discriminant is a perfect square (e.g. 25, 49, 121) or a fraction made of perfect squares (e.g. $\frac{1}{4}$, $\frac{9}{64}$), the roots are **rational**.
- Otherwise the roots are **irrational**.

<u>Example 1</u>

For the quadratic equation $x^2 - 4x - 7$, are the roots:
(a) Real? (b) Rational?

Solution

For this equation $a = 1$, $b = -4$ and $c = -7$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-4)^2 - 4 \times 1 \times (-7) \\ &= 16 - (-28) \\ &= 44\end{aligned}$$

(a) The discriminant is 44, which is positive, so yes the roots are real.
(b) 44 is not a perfect square, so no, the roots are not rational.

Sometimes equations must be rearranged into the standard form $ax^2 + bx + c$ first. Occasionally they may not even look like a quadratic equation at first.

Example 2 – requires rearranging

Find the nature of the roots of the equation $\frac{2x-3}{3x+4} = x$.

Solution

Rearranging first:

$$\frac{2x-3}{3x+4} = x$$

$$2x - 3 = x(3x+4) \quad \text{(multiplying by } (3x+4)\text{)}$$

$$2x - 3 = 3x^2 + 4x \quad \text{(expanding the bracket)}$$

$$3x^2 + 4x - 2x + 3 = 0 \quad \text{(rearranging to get 0 on the RHS)}$$

$$3x^2 + 2x + 3 = 0$$

So for this equation, $a = 3$, $b = 2$ and $c = 3$

$$\Delta = b^2 - 4ac$$
$$= 2^2 - 4 \times 3 \times 3$$
$$= 4 - 36$$
$$= -32$$

The discriminant is –32, which is less than zero, so this equation has <u>no real roots</u>.

Other occasions, you may be told the nature of the roots for an equation with an unknown constant in it, and you have to work out what possible range values may be used in the equation.

Example 3

The graph of the function $f(x) = x^2 + 2x + p$ does not touch or cross the x-axis. What is the range of values for p?

Solution

If the graph does not touch or cross the x-axis, this means that the function has no real roots, i.e. $\Delta < 0$.

For this equation, $a = 1$, $b = 2$ and $c = p$:

$$\Delta = b^2 - 4ac$$
$$= 2^2 - 4 \times 1 \times p$$
$$= 4 - 4p$$

For there to be no real roots, we need $\Delta < 0$, i.e.:

$$4 - 4p < 0$$
$$4p > 4$$
$$p > 1$$

Sometimes when there is a constant in the equation, you have to be very careful when rearranging, as you have to distinguish between different letters.

Example 4 – harder

If $\dfrac{(x-2)^2}{x^2+2}=k, \quad k\in\mathbb{R}$, find the values of *k* such that the given equation has two equal roots.

Solution

First we must rearrange:

$$\frac{(x-2)^2}{x^2+2}=k$$
$$(x-2)^2=k(x^2+2)$$
$$x^2-4x+4=kx^2+2k$$
$$kx^2+2k-x^2+4x-4=0$$

Now we must identify *a*, *b* and *c*. It is easy to get confused because there are '*k*'s and '*x*'s all mixed together. We have to remember that *k* is a fixed number (though we don't know it yet), and *x* is the variable.

We need to carefully split up the equation to put the terms with x^2 in first, then those in *x* and finally those without *x*.

$$kx^2+2k-x^2+4x-4=0$$
$$kx^2-x^2 \qquad +4x \qquad +2k-4=0$$
$$(k-1)x^2 \qquad +4x \qquad +(2k-4)=0 \qquad \text{(factorising to isolate } x^2\text{, } x \text{ and constants)}$$

We can now see that $a=(k-1)$, $b=4$ and $c=2k-4$. Now we work out the discriminant:

$$\begin{aligned}\Delta &= b^2-4ac\\ &=4^2-4(k-1)(2k-4)\\ &=16-4(2k^2-6k+4)\\ &=16-8k^2+24k-16\\ &=24k-8k^2\end{aligned}$$

The question tells us that we require equal roots, so we need $\Delta=0$.

$$24k-8k^2=0$$
$$8k(3-k)=0 \qquad \text{(taking out common factor } 8k\text{)}$$
$$8k=0 \qquad 3-k=0$$
$$k=0 \qquad k=3$$

Answers: $k = 0$ and $k = 3$.

Example 5 – proof

Show that the roots of $kx^2+3x+3=k$ are real for all values of *k*.

Solution

$$kx^2+3x+3=k$$
$$kx^2+3x+(3-k)=0 \quad \text{(moving } k \text{ to the LHS)}$$

Therefore $a = k$, $b = 3$ and $c = (3 - k)$:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= 3^2 - 4(k)(3-k) \\
&= 9 - 4k(3-k) \\
&= 9 - 12k + 4k^2 && \text{(simplifying)} \\
&= 4k^2 - 12k + 9 && \text{(rearranging into standard order)} \\
&= (2k-3)(2k-3) && \text{(factorising)} \\
&= (2k-3)^2
\end{aligned}$$

Therefore the discriminant is a perfect square. A perfect square cannot be negative (squaring always gives a positive answer), so the roots are always real and the discriminant cannot be negative.