Relationships and Calculus Unit

Factorising Polynomials and Solving Equations

A **polynomial** is a sum of powers of *x*. The **degree** of the polynomial refers to the highest power of *x* in the polynomial. For example:

- $3x^2 - 4x + 2$ is a polynomial of degree 2, also known as a **quadratic**.
- $5x^3 - 4x^2 + 7x - 1$ and $y^3 - 11x + 2$ are polynomials of degree 3, known as **cubics**.
- $x^4 + 3x^2 + 1$ is a polynomial of degree 4, also known as a **quartic**.
- $3a^5 - 2a + 6$ is a polynomial of degree 5.

The **coefficient** of a term in the polynomial is the number in front of the letter. e.g. in $x^4 + 2x^2 - 7x$, the coefficient of *x* is –7, the coefficient of x^2 is 2 and the coefficient of x^4 is 1.

In this unit, we work mostly with cubic expressions, focussing on factorising them and solving cubic equations, and sketching their graphs. The general form of a cubic is $ax^3 + bx^2 + cx + d$.

Synthetic Division

Synthetic division is a technique for evaluating polynomials quickly. It involves setting up a division grid like the one on the right. The diagram shows the basic grid for evaluating the polynomial $f(x) = ax^3 + bx^2 + cx + d$ when the input number is '$x = n$'.

$$\begin{array}{c|cccc} & x^3 & x^2 & x & c \\ n & a & b & c & d \\ & \downarrow & & & \\ \hline & & & & \square \end{array}$$

Example 1

$$\mathbf{g(x) = 4x^3 + 5x^2 - x - 22}. \text{ Evaluate } \mathbf{g(2)}.$$

Solution

Step one: set up the table with the coefficients and the value we are evaluating (the 'input number') in the top row.

$$\begin{array}{c|cccc} & x^3 & x^2 & x & c \\ 2 & 4 & 5 & -1 & -22 \\ & \downarrow & & & \\ \hline & & & & \square \end{array}$$

Step two: 'bring down' the number in the left column.

$$\begin{array}{c|cccc} 2 & 4 & 5 & -1 & -22 \\ & \downarrow & & & \\ \hline & 4 & & & \square \end{array}$$

Step three: multiply the number we have brought down by the input number and write it under the next coefficient.

$$\begin{array}{c|cccc} 2 & 4 & 5 & -1 & -22 \\ & \downarrow & 8 & & \\ \hline & 4 & & & \square \end{array}$$

Step four: add the two numbers in the column together, and then multiply the total by the input number.

$$\begin{array}{c|cccc} 2 & 4 & 5 & -1 & -22 \\ & \downarrow & 8 & 26 & \\ \hline & 4 & 13 & & \square \end{array}$$

Repeat step four (bring down and multiply, bring down, multiply bring down…) until the final column is complete. The answer is the number in the final box.

$$\begin{array}{c|cccc} 2 & 4 & 5 & -1 & -22 \\ & \downarrow & 8 & 26 & 50 \\ \hline & 4 & 13 & 25 & \boxed{28} \end{array}$$

Answer: $g(2) = 28$. You could check this by evaluating the function the 'normal' way.

If there is no x^2 or x term, we must still insert '0' in the relevant column. We cannot miss the column out.

Example 2

$f(x)=3x^3-4x-7$. Evaluate $f(-2)$.

$$\begin{array}{r|rrrr} & x^3 & x^2 & x & c \\ -2 & 3 & 0 & -4 & -7 \\ & \downarrow & & & \\ \hline & & & & \boxed{} \end{array}$$

Solution

Step one: set up the table with the coefficients and the value we are evaluating. **Important:** because there is no x^2 term, we have to insert 0 in the x^2 column. We cannot just miss the column out!

Steps two to four: complete the table using the method described in Example 1, being careful with the rules of integers.

$$\begin{array}{r|rrrr} -2 & 3 & 0 & -4 & -7 \\ & \downarrow & -6 & 12 & -16 \\ \hline & 3 & -6 & 8 & \boxed{-23} \end{array}$$

Answer: $f(-2)=-23$.

Synthetic division has two uses. One is to evaluate polynomials (as shown in the previous example), the other use is to divide polynomials. To divide by $(x-a)$, we use a as the input number. The number in the box at the end is the **remainder** after the division.

Example 3

Calculate the remainder when x^3-3x^2-4x+7 is divided by $(x-3)$.

Solution

Step one: set up the table as normal, using 3 as the input number (because 3 is the root of the bracket $(x-3)$).

$$\begin{array}{r|rrrr} 3 & 1 & -3 & -4 & 7 \\ & \downarrow & & & \\ \hline & & & & \boxed{} \end{array}$$

Steps two to four: complete the table using the method outlined in Example 1.

$$\begin{array}{r|rrrr} 3 & 1 & -3 & -4 & 7 \\ & \downarrow & 3 & 0 & -12 \\ \hline & 1 & 0 & -4 & \boxed{-5} \end{array}$$

Answer: the remainder is <u>–5</u>.

To divide by $(ax-b)$, we use $\frac{b}{a}$ as the divisor (because $\frac{b}{a}$ is the root of the bracket $(ax-b)$).

Example 4

Calculate the remainder when $2x^3-4x^2+6x+1$ is divided by $(2x+1)$.

Solution

Step one: set up the table as normal, using $-\frac{1}{2}$ as the input number (because $-\frac{1}{2}$ is the root of the bracket $(2x+1)$).

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -4 & 6 & 1 \\ & \downarrow & & & \\ \hline & & & & \boxed{} \end{array}$$

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(Example 4 continued)

Steps two to four: complete the table using the method outlined in Example 1.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -4 & 6 & 1 \\ & \downarrow & -1 & 2\cdot5 & -4\cdot25 \\ \hline & 2 & -5 & 8\cdot5 & \boxed{-3\cdot25} \end{array}$$

Answer: the remainder is $\underline{-3\cdot25}$.

When we use synthetic division, we can use it to write the original function $f(x)$, in the form $f(x) = (x-h)Q(x) + R$, where R is the remainder. In this expression, $Q(x)$ is a polynomial (called the quotient) whose coefficients come from the bottom row of the synthetic division table.

Example 5

Express the polynomial $3x^3 - 4x^2 + 9x - 1$ in the form $(x-2)Q(x) + R$.

Solution

We perform synthetic division on the cubic. We use 2 as the input number (because 2 is the root of the bracket $(x-2)$.

$$\begin{array}{r|rrrr} 2 & 3 & -4 & 9 & -1 \\ & \downarrow & 6 & 4 & 26 \\ \hline & 3 & 2 & 13 & \boxed{25} \end{array}$$

This shows that the remainder is 25.

The coefficients of the quotient are read from the bottom line of the table (ignoring the remainder). These are 3, 2 and 13, which means that $Q(x) = 3x^2 + 2x + 13$.

Answer: $(x-2)(3x^2 + 2x + 13) + 25$.

Factorising Cubics

If the final number (in the box) in a synthetic division (by *a*) is zero, then it means that:

- $f(a) = 0$, meaning that *a* is a **root** of the equation.
- The remainder on dividing by $(x-a)$ is 0, hence $(x-a)$ is a **factor** of the polynomial.
- The quotient from the bottom row of the synthetic division table is also a factor.

The Factor and Remainder Theorem

If $f(x)$ is divided by $(x-h)$ and the remainder is $f(h)$, then

$$f(h) = 0 \Leftrightarrow (x-h) \text{ is a factor of } f(x)$$

This means that we can use synthetic division to help identify factors of a polynomial.

Example 1

Show that $(x-5)$ is a factor of $x^3 + 2x^2 - 23x - 60$.

Solution

$(x-5)$ is a factor $\Leftrightarrow$ 5 is a root, so we use 5 as the input number and complete a synthetic division table, as shown.

The remainder is zero, therefore $(x-5)$ is a factor. ■

$$\begin{array}{r|rrrr} 5 & 1 & 2 & -23 & -60 \\ & \downarrow & 5 & 35 & 60 \\ \hline & 1 & 7 & 12 & \boxed{0} \end{array}$$

Fact: if synthetic division shows that $(x-a)$ is a factor, then the coefficients of the other factor are given by the quotient in the bottom row of the table (the numbers we 'brought down').

In the example above, the other factor is $x^2 + 7x + 12$ (or $(x+3)(x+4)$ in factorised form).

Exam questions will often begin with a part (a) that asks you to 'show that' a value is a root, or a bracket is a factor. You must clearly state the conclusion in one of the following forms:

If the question asked you to…	**Then your conclusion should say…**
Show that $x = a$ is a root of…	Remainder is 0, therefore $x = a$ is a factor.
Show that $(x - a)$ is a factor of…	Remainder is 0, therefore $(x - a)$ is a factor.

You would lose a mark is you used the word 'root' when you meant 'factor' or vice versa. Roots are numbers (of the form $x = a$). Factors are brackets.

Example 2

(a) Show that $x = 1$ is a root of $x^3 + 6x^2 - x - 6$.

(b) Hence, factorise $x^3 + 6x^2 - x - 6$ fully.

Solution

(a) Performing synthetic division using 1 as the input number, we get the table shown on the right:

$$\begin{array}{r|rrrr} 1 & 1 & 6 & -1 & -6 \\ & \downarrow & 1 & 7 & 6 \\ \hline & 1 & 7 & 6 & \boxed{0} \end{array}$$

The remainder is zero, hence $x = 1$ is a root. ■

(this <u>must</u> be stated explicitly)

(b) Using the bottom line of the table, we see that the other factor is $x^2 + 7x + 6$. Therefore:

$$\begin{aligned} x^3 + 6x^2 - x - 6 &= (x-1)(x^2 + 7x + 6) \\ &= \underline{\underline{(x-1)(x+1)(x+6)}} \end{aligned}$$

Normally an exam question will tell you one root to help you get started. If you are not told a root, and are just asked to factorise, then you have to experiment to find the first root.

To factorise $ax^3 + bx^2 + cx + d$, keep on trying different factors of d until you find one that has a remainder of zero (start with 1, then –1, then 2, then –2 and so on) until you find one that has a remainder of zero.

Example 3 – requiring experimentation

Factorise $2x^3 - 15x^2 + 16x + 12$ fully.

Solution

We experiment until we find a factor that gives a remainder of zero. We try all the factors of 12, so we try $\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12$ in order.

First try $x = 1$:

$$\begin{array}{r|rrrr} 1 & 2 & -15 & 16 & 12 \\ & \downarrow & 2 & -13 & 3 \\ \hline & 2 & -13 & 3 & \boxed{15} \end{array}$$

Remainder is 15, so 1 is **NOT** a root.

Now try $x = -1$:

$$\begin{array}{r|rrrr} -1 & 2 & -15 & 16 & 12 \\ & \downarrow & -2 & 17 & -33 \\ \hline & 2 & -17 & 33 & \boxed{-21} \end{array}$$

Remainder is –21, so –1 is **NOT** a root.

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(Example 3 continued)

Now try x = 2:

$$\begin{array}{r|rrrr} 2 & 2 & -15 & 16 & 12 \\ & \downarrow & 4 & -22 & -12 \\ \hline & 2 & -11 & -6 & \boxed{0} \end{array}$$

Remainder is 0, so 2 **IS** a root.

Now we have found a root, we can complete the factorising in the normal way:

$$\begin{aligned} 2x^3 - 15x^2 + 16x + 12 &= (x-2)(2x^2 - 11x - 6) \\ &= (x-2)(2x+1)(x-6) \end{aligned}$$

Example 4 – finding an unknown coefficient

Given that $(x+2)$ is a factor of $2x^3 + x^2 + kx + 2$, calculate k.

Solution

If $(x+2)$ is a factor, then –2 is a root. We complete a synthetic division using '–2' as the input number:

$$\begin{array}{r|rrrr} -2 & 2 & 1 & k & 2 \\ & \downarrow & -4 & 6 & -2k-12 \\ \hline & 2 & -3 & k+6 & \boxed{-2k-10} \end{array}$$

The remainder is $-2k-10$. However since we are told $(x+2)$ is a factor, we know this remainder must also equal zero. Therefore we can solve an equation>

$$\begin{aligned} -2k - 10 &= 0 \\ 2k &= -10 \\ k &= -5 \end{aligned}$$

Solving Cubic Equations

To solve a cubic equation, it must first be factorised, using the method from the last section.

Example

(a) Show that $x = 1$ is a root of $x^3 + 6x^2 + 3x - 10$.

(b) Hence, solve the equation $x^3 + 6x^2 + 3x - 10 = 0$.

Solution

(a) Performing synthetic division using 1 as the input number, we get:

$$\begin{array}{r|rrrr} 1 & 1 & 6 & 3 & -10 \\ & \downarrow & 1 & 7 & 10 \\ \hline & 1 & 7 & 10 & \boxed{0} \end{array}$$

The remainder is zero, hence $x = 1$ is a root. ■

(b) To solve the equation, we use the synthetic division table from part (a) to factorise.

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(Example continued)

Using the table, we can see from the numbers '1, 7, 10' in the bottom row that the other factor is $x^2+7x+10$. Therefore:

$$x^3+6x^2+3x-10=0$$

$$(x-1)(x^2+7x+10)=0$$

$$(x-1)(x+5)(x+2)=0$$

$$\downarrow \qquad \downarrow \qquad \searrow$$

$$\underline{\underline{x=1,\ x=-5,\ x=-2}}$$

Determining the Equation of a Polynomial from its roots

If all of the roots of a polynomial are real, we can work out its equation if we are told (or shown via a graph) all of its roots and at least one other point on the parabola.

This is because if we are told that one of the roots of the parabola is $x=a$, then we know that $(x-a)$ will be a bracket in the factorised form of the parabola.

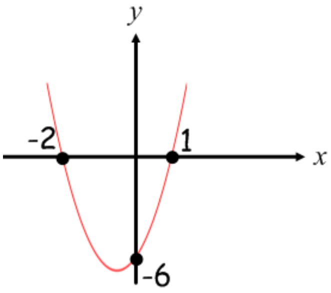
Fact

If we are told that the roots of a quadratic expression are a and b, then the quadratic must be of the form $y=k(x-a)(x-b)$, for some value of k that needs to be found.

If we are told that the roots of a cubic expression are a, b and c, then the cubic must be of the form $y=k(x-a)(x-b)(x-c)$, for some value of k that needs to be found.

Example 1 – quadratic

Determine the equation of the parabola shown in the diagram.



Solution

The roots are –2 and 1, so the parabola has an equation of the form -.

We are told that the parabola goes through the point (0, –6), so we substitute x = 0 and y = –6 into our equation:

$$y=k(x+2)(x-1)$$

$$-6=k(0+2)(0-1) \quad \text{(substituting in } x=0 \text{ and } y=-6\text{)}$$

$$-6=k(2)(-1)$$

$$-6=-2k \quad \text{(simplifying)}$$

$$k=\frac{-6}{-2}$$

$$k=3$$

Answer: the parabola has equation $y=3(x+2)(x-1)$

The method works in the same way for cubic graphs, except you will need three brackets, so the equation will be of the form $y=k(x-a)(x-b)(x-c)$ for roots a, b and c.

Example 2 – cubic

A cubic function has roots 2, 4 and –1 and its graph goes through the point (3, 8). Determine the equation of the cubic.

Solution

The roots we are told tell us that the equation is of the form $y = k(x-2)(x-4)(x+1)$.

Since we are also told the point (3, 8) is on the graph, we substitute x = 3 and y = 8 into this equation:

$$\begin{aligned}
&y = k(x-2)(x-4)(x+1)\\
&8 = k(3-2)(3-4)(3+1) \quad \text{(substituting in } x=3 \text{ and } y=8)\\
&8 = k(1)(-1)(4)\\
&8 = -4k \qquad\qquad \text{(simplifying)}\\
&k = \frac{8}{-4}\\
&k = -2
\end{aligned}$$

Answer: the cubic has equation $y = -2(x-2)(x-4)(x+1)$

A **repeated root** occurs when exactly the same bracket appears twice in the fully factorised expression. This will be shown by the bracket being squared. For example in the equations $y=(x-4)^2$ and $y=(x+5)(x+3)^2$ there are repeated roots (4 in the first equation, –3 in the second equation).

A repeated root can be recognised from a diagram since it looks different to a 'regular' root:

- At a 'regular' root, the graph of the equation 'passes through' the x-axis (for example the root at x = 2 in the diagram for Example 3).
- At a repeated root, the graph is tangent to the x-axis. For example, in the diagram for Example 3 below, the root at x = –3 is a repeated root.

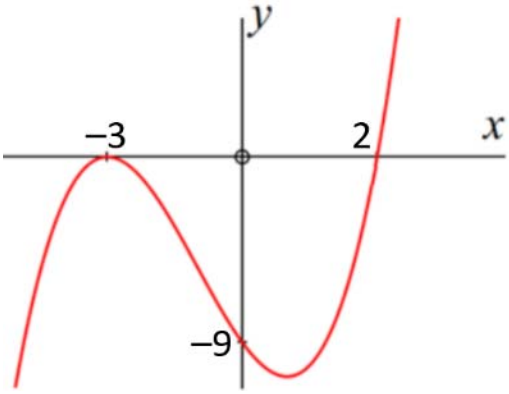
A repeated root is an exampleof a stationary point (see page 103).

Example 3 – graph with a repeated root

Determine the equation of the cubic graph shown in the diagram.

Solution

The curve is tangent to the x axis at –3. This means that x = –3 is a repeated root, which means its bracket is 'squared'.



$$\begin{aligned}
y &= k(x+3)(x+3)(x-2)\\
&= k(x+3)^2(x-2)
\end{aligned}$$

We also know the the graph goes through the point (0, –9), so we substitute x = 0 and y = –9 into this equation:

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