Vector Pathways

We can use the rules of adding and taking away vectors to express a vector $\overrightarrow{AB}$ in a geometrical situation as a combination of other, known, vectors.

To do this, we identify a route, or pathway, between A and B, in which each step of the route can be expressed in terms of one of the other known pathways. We can choose any route we like, and the final answer, when simplified, will always be the same.

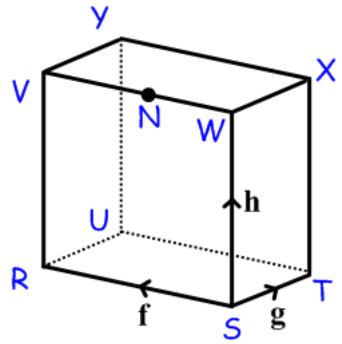
Example 4 – vector pathways (adapted from 2008 SQA exam question)

In the diagram, RSTUVWXY represents a cuboid.

$\overrightarrow{SR}$ **represents vector f, $\overrightarrow{ST}$ represents vector g and $\overrightarrow{SW}$ represents vector h.**

N is the midpoint of VW.

Express $\overrightarrow{RX}$ and $\overrightarrow{TN}$ in terms of f, g and h.



Solution

For $\overrightarrow{RX}$:

Step one – identify a pathway from R to X.

One possible pathway is $\overrightarrow{RU}, \overrightarrow{UY}, \overrightarrow{YX}$

Step two – express each part of the pathway in terms of a known vector

$\overrightarrow{RU} = \mathbf{g}$, $\overrightarrow{UY} = \mathbf{h}$, $\overrightarrow{YX} = \text{backwards along } \mathbf{f}$

Therefore $\overrightarrow{RX} = \mathbf{g} + \mathbf{h} - \mathbf{f}$ (or $-\mathbf{f} + \mathbf{g} + \mathbf{h}$ or any other algebraically equivalent expression)

For $\overrightarrow{TN}$:

Step one – identify a pathway from T to N:

One possible pathway is $\overrightarrow{TX}, \overrightarrow{XW}, \overrightarrow{WN}$

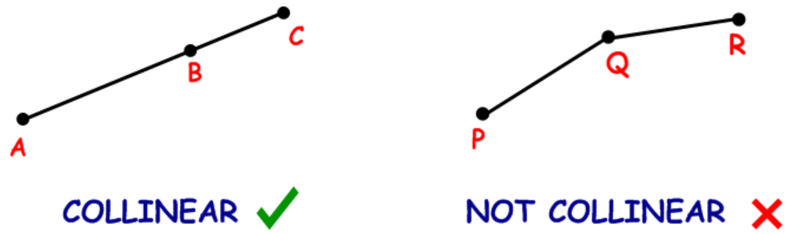
Step two – express each part of the pathway in terms of a known vector:

$\overrightarrow{TX} = \mathbf{h}$, $\overrightarrow{XW} = \text{backwards along } \mathbf{g}$, $\overrightarrow{WN} = \text{half way along } \mathbf{f}$

Therefore $\overrightarrow{TN} = \mathbf{h} - \mathbf{g} + \frac{1}{2}\mathbf{f}$ (or $\frac{1}{2}\mathbf{f} - \mathbf{g} + \mathbf{h}$ or any other algebraically equivalent expression)

Collinear Points and Ratio

Three or more points are said to be **collinear** if they lie on the same straight line:



In two dimensions:

- If three points A, B and C are collinear, then AB, AC and BC all have the **same gradient**.

- If three points P, Q and R are not collinear, then PQ, QR and PR will all have **different gradients**.

To show whether three points A, B and C are collinear, we must therefore show that AB and BC have the same gradient. If they have the same gradient, and since B lies on both line segments, we can conclude that the lines are collinear. [**note:** AB and BC was a random choice. We could use AB/AC or AC/BC instead, so long as the two line segments have a common point].

For a sufficient conclusion at the end of a collinearity proof, it is essential to use the words **'parallel'** and **'common point'**.

Example 1 – two dimensions

Show that the points P(–6,–1), Q(0,2) and R(8,6) are collinear.

Solution

$$m_{PQ} = \frac{2-(-1)}{0-(-6)} = \frac{3}{6} = \frac{1}{2}$$

$$m_{QR} = \frac{6-2}{8-0} = \frac{4}{8} = \frac{1}{2}$$

The two line segments are parallel and Q is a common point, therefore P, Q and R are collinear.

As stated for two dimensions above, three points are said to be **collinear** if they lie on the same straight line. This applies in three dimensional space as well.

If three points A, B and C are collinear, then the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are all parallel (recall that two vectors **u** and **v** are parallel if $\mathbf{u} = k\mathbf{v}$ for some $k \neq 0$).

To show whether three points P, Q and R are collinear, we must therefore show that $\overrightarrow{PQ} = k\overrightarrow{QR}$ for some $k \neq 0$. It is crucial that the two vectors have a common point (in this case Q), otherwise it would just mean that they were parallel and not collinear. [**note:** we could just as well use $\overrightarrow{PQ}$ & $\overrightarrow{PR}$ or $\overrightarrow{QR}$ & $\overrightarrow{PR}$, so long as the two vectors have a common point].

Example 2 – three dimensions

Prove that A(3, 4, 1), B(9, 1, –5) and C(11, 0, –7) are collinear.

Solution

First find $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

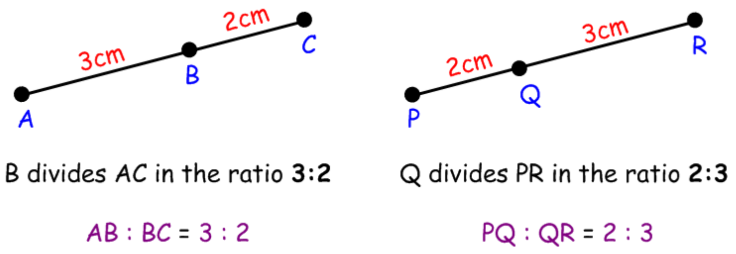
$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix}$$

$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} 11 \\ 0 \\ -7 \end{pmatrix} - \begin{pmatrix} 9 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

$\therefore \overrightarrow{AB} = 3\overrightarrow{BC}$, so $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel.

B is a common point, so A, B and C are collinear. ■

When three points are collinear, they form a line which is divided internally by the middle point in a ratio. Order matters, as shown in the diagram on the next page.

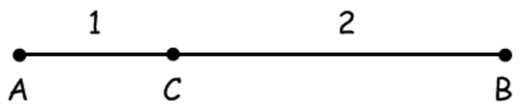


Note: A midpoint divides a line internally in the ratio 1:1.

Example 3 – coordinates given ratio

A is the point (3, –2, 4) and B is (3, 4, –1). The point C divides AB internally in the ratio 1:2. Find the coordinates of C.

Solution



Start with a sketch to illustrate the situation. This will prevent you from getting confused about which order the letters go in.

Setting this up as a ratio gives us $\dfrac{\overrightarrow{CB}}{\overrightarrow{AC}} = \dfrac{2}{1}$, or $\overrightarrow{CB} = 2\overrightarrow{AC}$, which we can now do some vector arithmetic on:

$$\begin{aligned}
2\overrightarrow{AC} &= \overrightarrow{CB} \\
2(\mathbf{c} - \mathbf{a}) &= \mathbf{b} - \mathbf{c} \\
2\mathbf{c} - 2\mathbf{a} &= \mathbf{b} - \mathbf{c} \\
2\mathbf{c} + \mathbf{c} &= 2\mathbf{a} + \mathbf{b} \\
3\mathbf{c} &= 2\mathbf{a} + \mathbf{b} \\
\mathbf{c} &= \frac{2\mathbf{a} + \mathbf{b}}{3}
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} &= \frac{2\mathbf{a} + \mathbf{b}}{3} \\
&= \frac{\begin{pmatrix} 6 \\ -4 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}}{3} \\
&= \frac{1}{3}\begin{pmatrix} 9 \\ 0 \\ 7 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 0 \\ \frac{7}{3} \end{pmatrix}
\end{aligned}$$

i.e. C is the point $(3, 0, \frac{7}{3})$.

Example 4 – ratio given co-ordinates

Given that the points A(–6, 3, –1), B(–18, –6, 14) and C(–26, –12, 24) are collinear, calculate the ratio in which B divides AC.

Solution

First find $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

$$\begin{aligned}
\overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\
&= \begin{pmatrix} -18 \\ -6 \\ 14 \end{pmatrix} - \begin{pmatrix} -6 \\ 3 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -12 \\ -9 \\ 15 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{BC} &= \mathbf{c} - \mathbf{b} \\
&= \begin{pmatrix} -26 \\ -12 \\ 24 \end{pmatrix} - \begin{pmatrix} -18 \\ -6 \\ 14 \end{pmatrix} \\
&= \begin{pmatrix} -8 \\ -6 \\ 10 \end{pmatrix} = \frac{2}{3}\overrightarrow{AB}
\end{aligned}$$

$\overrightarrow{BC} = \frac{2}{3}\overrightarrow{AB}$

$\frac{\overrightarrow{BC}}{\overrightarrow{AB}} = \frac{2}{3}$ (i.e. $\overrightarrow{AB}$ corresponds to 3 and $\overrightarrow{BC}$ corresponds to 2)

$\therefore \overrightarrow{AB} : \overrightarrow{BC} = 3:2$

i.e. B divides AC in the ratio 3:2