# Topic 1·2: Sequences and Series

A **sequence** is an ordered list of numbers, e.g. 1, 2, 4, 8, 16, … or $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.
A **series** is a sum of the terms of a sequence, e.g. 1 + 2 + 4 + 8 + 16 + … or $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$.

## Sigma Notation

The notation $\sum_{k=a}^{b} f(k)$ is shorthand for the series $f(a) + f(a+1) + f(a+2) + \ldots + f(b)$, where $a$ and $b$ are integers such that $a \le b$.

e.g.

$$\begin{aligned}\sum_{k=1}^{3}(2k+1) &= (2\times1+1)+(2\times2+1)+(2\times3+1)\\ &= 3+5+7\\ &= 15\end{aligned}$$

$$\begin{aligned}\sum_{r=0}^{4}(-2)^r &= (-2)^0+(-2)^1+(-2)^2+(-2)^3+(-2)^4\\ &= 1+(-2)+4+(-8)+16\\ &= 11\end{aligned}$$

$$\begin{aligned}\sum_{i=2}^{5}(3i^2-7) &= (3\times2^2-7)+(3\times3^2-7)+(3\times4^2-7)+(3\times5^2-7)\\ &= 5+20+41+68\\ &= 134\end{aligned}$$

## Arithmetic Sequences

An arithmetic sequence is one with a common difference (i.e. where the next term is obtained by adding or subtracting a constant number to the previous term, as in the recurrence relation $u_{n+1} = u_n + d$), such as 1, 3, 5, 7, … or 4, 1·5, −1, −3·5, ….

**Notation:** A general arithmetic sequence has first term $a$ and common difference $d$. The $n^{th}$ term is denoted by $u_n$. If $d > 0$ then the sequence is increasing, if $d < 0$ then it is decreasing.

The final formula (for the partial sum) is given on the formulae sheet in the exam and in unit assessments, and the other formulae are not given.

**Formulae**
For an arithmetic sequence,
- $d$ can be obtained by subtracting two consecutive terms.
- the $n^{th}$ term is given by the formula $u_n = a + (n-1)d$
- and the 'partial sum' of the first $n$ terms is given by the formula $S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$

Example 1

**For the arithmetic sequence 7, 10, 13, 16, …find**
**(a) the 27$^{th}$ term, and**
**(b) a formula for the $n^{th}$ term.**

**Solution**

By inspection the first term $a = 7$, and the common difference $d = 3$.

Using the formula $u_n = a + (n-1)d$ gives:

(a) $u_{27} = a + 26d$
$= 7 + 26 \times 3$
$= 85$

(b) $u_n = a + (n-1)d$
$= 7 + 3(n-1)$
$= 7 + 3n - 3$
$= 3n + 4$

Sometimes you are not told the first term, or even two c[illegible]utive terms, but as long as you are given any two terms, you can work these out.

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Example 2

**In an arithmetic sequence, $u_3 = 60$ and $u_5 = 50$. [illegible]**
**(a) The first term and the com[illegible] difference**
**(b) The 15th term of the sequenc[illegible]**
**(c) The sum of the first 6[illegible]s**

**Solution**

(a) Using the formula [illegible]$n-1)d$, we have $u_3 = a + 2d$ [illegible]nd $u_5$ [illegible]
meaning th[illegible] $u_3$ [illegible] hopefully this result should [illegible]e when you think about it)

Th[illegible], so $d = -5$.

[illegible]
$60 =$ [illegible]$2d$
[illegible]$= a$ [illegible]$10$
[illegible]$70$

[illegible] the first term is 70 and th[illegible] difference is $-5$.

(b) $u_{15} = a + 14d$
$= 70 + 14 \times (-5)$
$= 0$

(c) $S_n = \frac{1}{[illegible]}$ [illegible]$[2a$ [illegible]$1)d]$

$S_{60} = \frac{1}{2}$ [illegible]$70 + 59 \times (-5)]$
$= 30(140 - 295)$
$= -4650$

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Exa[illegible] 3

**[illegible]ind the sum of the first 18 terms of the arithmetic series $3 + 9 + 15 + 21 + ...$**

**[illegible]tion**

On inspection, we see that $a = 3$ and $d = 6$.

Using these values in the formula $S_n = \frac{1}{2}n[2a + (n-1)d]$ gives us: $S_{18} = \frac{18}{2}[2(3) + 17(6)]$
$= 9(6 + 102)$
$= 972$

The sum of the first 18 terms is 972.

Example 4

**Find the sum of the arithmetic series $12 + 19 + 26 + 33 + ... + 285$.**

**Solution**

By inspection, $a$ = 12 and $d$ = 7. We must find $n$.

We know that $u_n = 285$, therefore

$$a + (n-1)d = 285$$
$$12 + (n-1)7 = 285$$

$12 + 7n - 7$ [illegible]

$7n$ [illegible] $= 28$[illegible]

[illegible], i.e. [illegible]re are 40 terms in the series

Now using the formula $S_n = \frac{1}{2}n\left[2\right.$[illegible]$\left.(n-1)d\right]$ $S$[illegible] $\frac{\text{[illegible]}0}{2}\left[2(12) + 39(7\right.$[illegible] :

$$= 20 \times 297$$
$$= 5940$$

The sum of the series [illegible]0

## Geometric Sequence[illegible]

A geometric s[illegible] with [illegible] common ratio (i.e. whe[illegible] next [illegible] is obtained by multiplying the p[illegible]us [illegible] a constant, as in the re[illegible]urrenc[illegible]tion $u_{n+1} = ru_n$), such as 1, 2, 4, 8, [illegible]… or 81, [illegible] 9, 3, 1, $\frac{1}{3}$,….

**Notati**[illegible] ge[illegible]equence has first term $a$ [illegible]d com[illegible]atio $r$. The $n^{\text{th}}$ term is denot[illegible] $u_n$.

If $|r| > 1$ t[illegible] the sequence is increasing[illegible] $|r$[illegible] the sequence is decrea[illegible] [illegible]the[illegible] all terms are either all [illegible]sitive [illegible]all negative (depending[illegible]he first[illegible] If $r < 0$, th[illegible]he terms alternate from po[illegible]e to [illegible]ive e.g. 1, –2, 4, –8, 16,[illegible]

[illegible] final formula (for the part[illegible] given on the formula sh[illegible] ex[illegible] and in unit assessments, and the o[illegible]rmulae a[illegible] not given.

**Formulae**

For a geometric sequ[illegible]

- $r$ can be obtained by [illegible]ing two consecutive terms[illegible]
- The $n$[illegible]rm is given by the formula $u_n = ar^{n-1}$
- A[illegible] 'partial sum' of the first $n$ terms is given by the [illegible]la

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad (\text{ or } S_n = \frac{a(r^n-1)}{r-1})$$

**(ei[illegible]version of this final formula can be used, though it makes for easier calculations to use the first one when $r < 1$ and the second one when $r > 1$)**

Example 1

**Find the 12th term of the geometric sequence 2, 8, 32, 128, ...**

**Solution**

On inspection, $a$ = 2 and $r = \frac{8}{2} = 4$.

$$u_n = ar^{n-1} \quad \text{Thus:}$$
$$u_{12} = ar^{11}$$
$$= 2 \times 4^{11}$$
$$= \underline{8\ 388\ 608}$$

Example 2

**A geometric sequence of <u>positive</u> terms has third term 18 and seventh term 1458. Find the fifth term of this sequence.**

**Solution**

$u_n = ar^{n-1}$

$u_3 = ar^2 = 18$ ...(1)

$u_7 = ar^6 = 1458$ ...(2)

$$\frac{ar^6}{ar^2} = \frac{1458}{18}$$

(2) ÷ (1) $\Rightarrow$ $r^4 = 81$

$r = \pm 3$

But we were told in the question [illegible] all terms are positive, so $r > 0$, hence $r = 3$.

Substituting into (1): $ar^2 = 18$ $\Rightarrow$ $a \times 3^2 = 18$

$9a = 18$

$a = 2$

$u_n = ar^{n-1}$ $\Rightarrow$ $u_5 = ar^4$

$= 2 \times 3^4$

$= \underline{\underline{162}}$

Example 3

**Find the sum of the first 9 terms of the geometric series [illegible] + 8 + 16 + 32 + ...**

**Solution**

[illegible], $a = 4$ and $r = 2$. Thus $S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{4(2^9 - 1)}{2 - 1}$

$= \frac{4(2^9 - 1)}{1}$

$= \underline{\underline{2044}}$

Example 4

**Evaluate $\sum_{k=1}^{20} (0 \cdot 9)^k$, giving your answer correct to [illegible] decimal places.**

**Solution**

$\sum_{k=1}^{20} (0 \cdot 9)^k = 0 \cdot 9 + (0 \cdot 9)^2 + (0 \cdot 9)^3 + \ldots + (0 \cdot 9)^{20}$

This is a geometric series of 20 terms with $a = 0 \cdot 9$ and $r = 0 \cdot 9$.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{20} = \frac{0 \cdot 9\left\{1 - (0 \cdot 9)^{20}\right\}}{1 - 0 \cdot 9}$$

$$= \frac{0 \cdot 9\left\{1 - (0 \cdot 9)^{20}\right\}}{0 \cdot 1}$$

$$= 7 \cdot 906 \text{ (3 d.p.)}$$

Example 5

**Evaluate the sum of the geometric series $4+20+100+...+62\,500$.**

**Solution**

By inspection, $a = 4$ and $r = 5$. However we must find $n$.

We know that $u_n = 62500$ $\Rightarrow$

$$ar^{n-1} = 62500$$
$$4\times 5^{n-1} = 62500$$
$$5^{n-1} = 15625$$

This equation can be solved by taking natural logarithms of both sides of the equation:

$$\ln(5^{n-1}) = \ln 15625$$
$$(n-1)\ln 5 = \ln 15625$$
$$n-1 = \frac{\ln 15625}{\ln 5}$$
$$n-1 = 6$$

$n = 7$, i.e. there are 7 terms in the series

Using $S_n = \frac{a(r^n-1)}{r-1}$ gives $S_7 = \frac{4(5^7-1)}{5-1}$

$$= \frac{4(5^7-1)}{4}$$
$$= 78\,124$$

## Infinite Geometric Series

If $-1<r<1$ then successive terms will get increasingly smaller. This means that the sum of the first $n$ terms approaches as a limit as $n\to\infty$. We say that this geometric series has a **sum to infinity**, denoted $S_\infty$.

The following formulae is [illegible] the formula sheet in the e[illegible]d in [illegible] assessments.

**Formulae**

The sum to infinity of a geometric series when $-1<r<1$ is given by $S_\infty = \frac{a}{1-r}$

Example 6

**Find the value of the sum to infinity of the geometric series $36+27+20\cdot 25+...$, explaining why it is exists.**

**Solution**

$a = 36$ and $r = \frac{27}{36} = \frac{3}{4}$

$-1<\frac{3}{4}<1$, so the geometric series has a sum to infinity.

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{36}{1-\frac{3}{4}} = \frac{36}{\frac{1}{4}}$$
$$= 144$$

i.e. the sum to infinity is 144

Example 2

**For the series $1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+...$**

**(a) State the first term and the common ratio.**

**(b) State the range of values of *x* for which a sum to infinity exists.**

**(c) Find an expression for the sum to infinity when *x* is in this range of values.**

**Solution**

(a) $a = 1$ and $r=\frac{x}{2}$

(b) For a sum to infinity to exist, the common ratio [illegible] must be between $-1$ and 1

$$-1<\frac{x}{2}<1$$
$$\therefore -2<x<2$$

(c) When $x$ takes these values

$$S_\infty = \frac{a}{1-r}$$

[illegible]

$$\frac{1}{[illegible]} = \frac{[illegible]}{2-x}$$

## Maclaurin series

A **power series**, is where we express a function as an infinite sum of the form

$a_0 + a_1x + a_2x^2 + a_3x^3 + ...$ [illegible] (or $\sum_{k=0}^{\infty} a_k x^k$ ).

All functions can also be represented as a Power series (also called **Maclaurin series**) in this way – though it is important to realise that they only converge (i.e. only work) for specific ranges of $x$ – though [illegible] functions this range is [illegible] real numbers.

The following formula is given on the formula sheet [illegible]m and in unit assessments.

**Formula:** The Maclaurin series for $f(x)$ is given by:

$$f(x) = f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+...+\frac{f^{(n)}(0)}{n!}x^n+.........=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$

Example 1

**Find the Maclaurin series for the function $f(x)=\tan^{-1}x$ up to and including the term in $x^3$. (note – this series only converges when $-1\le x\le 1$ ).**

**Solution**

$f(x)=\tan^{-1}x$, so $f(0)=\tan^{-1}0=0$

$f'(x)=\frac{1}{1+x^2}$ (as learnt in the first unit), so $f'(0)=\frac{1}{1+0^2}=1$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \quad \text{(using chain rule)}, \quad \text{so } f''(0) = \frac{0}{(1+0^2)^2} = 0$$

$$f'''(x) = \frac{2(3x^2-1)}{(1+x^2)^3} \quad \text{(using quotient rule and simplifying)}, \quad \text{so } f'''(0) = \frac{2(0-1)}{(1+0)^3} = -2$$

Then using the Maclaurin series formula,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$= \cancel{0} + \frac{1}{1!}x + \cancel{\frac{0}{2!}x^2} + \frac{-2}{3!}x^3 + \ldots$$

$$= \frac{1}{1}x + \frac{-2}{6}x^3 + \ldots$$

$$= x - \frac{1}{3}x^3 + \ldots$$

It is also **very** useful to know a few standard power series, as knowing them and being able to just write them down can make some questions quicker.

The following formulae are not given on the formula sheet in the exam and in unit assessments.

**Formulae:** standard Maclaurin series

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$ converges for all $x \in \mathbb{R}$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdot$ converges for all $x \in \mathbb{R}$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdot$ converges for all $x \in \mathbb{R}$.

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ converges for $|x| \le 1$.

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots\cdot$ converges for $|x| < 1$.

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots\cdot$ converges for $|x| < 1$.

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdot$ converges for $-1 < x \le 1$.

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdot$ converges for $-1 \le x < 1$.

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Example 2

**Find the Maclaurin series for the function $f(x) = e^{2x}$ up to and including the term in $x^4$.**

**Solution**

There are two methods:

Method (a) – from first principles

Method (b) – memorise the series for $e^x$ and use that.

**Method (a)**

$f(x) = e^{2x}$, so $f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$, so $f'(0) = 2e^0 = 2$

$f''(x) = 4e^{2x}$, so $f''(0) = 4e^0 = 4$

$f'''(x) = 8e^{2x}$, so $f'''(0) = 8e^0 = 8$

$f^4(x) = 16e^{2x}$, so $f^4(0) = 16e^0 = 16$

Then using the Maclaurin series formula,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$= 1 + \frac{2}{1!}x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \ldots$$

$$= 1 + \frac{2}{1}x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \frac{16}{24}x^4 + \ldots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \ldots$$

**Method (b)**

We know that if $f(x) = e^x$, then $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

For $e^{2x}$ we use the series for $f(2x)$ - i.e. we replace every occurrence of *x* with 2*x*.

$$e^{2x} = f(2x)$$

$$= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

$$= 1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{6} + \frac{16x^4}{24} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

Which is the same result as for method (a).

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Example 3

**Obtain the Maclaurin series for the function $f(x) = e^x \sin x$ up to and including the term in $x^4$.**
**Hence write down the Maclaurin series for the function $g(x) = e^{2x} \sin 2x$.**

**Solution**

(1) **Method (a)** – from first principles

All derivatives are obtained using the product rule, but the working has been left out, but only to keep this example shorter. **You would be expected to show full working in an exam situation though.**

$f(x) = e^x \sin x$, so $f(0) = e^0 \sin 0 = 0$

$f'(x) = e^x(\sin x + \cos x)$, so $f'(0) = e^0(\sin 0 + \cos 0) = 1$

$f''(x) = 2e^x \cos x$, so $f''(0) = 2e^0 \cos 0 = 2$

$f'''(x) = 2e^x(\cos x - \sin x)$, so $f'''(0) = 2e^0(\cos 0 - \sin 0) = 2$

$f^4(x) = -4e^x \sin x$, so $f^4(0) = -4e^0 \sin 0 = 0$

Then using the Maclaurin series formula,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$= \not{0} + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \cancel{\frac{0}{4!}x^4} + \ldots$$

$$= x + \frac{2}{2}x^2 + \frac{2}{6}x^3 + \ldots$$

$$= x + x^2 + \frac{1}{3}x^3 + \ldots$$

Part (1) **Method (b)** – use known series for $e^x$ and $\sin x$

We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

This means that:

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots\right)$$

We can multiply the brackets together by:

- multiplying everything in the second bracket by 1 stopping when we first get a power larger than $x^4$ (because the question asked up to $x^4$).
- then multiplying everything in the second bracket by $x$ stopping when we first get a power larger than $x^4$.

then multiplying everything in the second bracket by $\frac{x^2}{2!}$ stopping when we first get a power larger than $x^4$ etc.

i.e.

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1\left(x - \frac{x^3}{3!}\right) + x\left(x - \frac{x^3}{3!}\right) + \frac{x^2}{2!}(x) + \frac{x^3}{3!}(x) + \ldots \text{ (all other terms give a power higher than } x^4\text{)}$$

$$= x - \frac{x^3}{3!} + x^2 - \cancel{\frac{x^4}{3!}} + \frac{x^3}{2!} + \cancel{\frac{x^4}{3!}} + \ldots$$

$$= x - \frac{x^3}{6} + x^2 + \frac{x^3}{2} + \ldots$$

$$= x + x^2 + \frac{x^3}{3} + \ldots \quad \text{(which is the same answer obtained using method (a))}$$

Part (2)

We know that $e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$

Replacing $x$ with $2x$ in this power series gives:

$$e^{2x} \sin 2x = 2x + (2x)^2 + \frac{1}{3}(2x)^3 + \ldots$$

$$= 2x + 4x^2 + \frac{8}{3}x^3 + \ldots$$