

Methods in Algebra and Calculus Unit

Topic 1·1: Partial Fractions

Algebraic Long Division

Long division is a written division algorithm where written subtraction sums are also used to calculate the remainders at each stage (instead of calculating them mentally). We use long division when we have an **improper** (also known as '**topheavy**') fraction where the numerator is greater than the denominator, such as $\frac{7}{4}$ or $\frac{23}{5}$.

It is not just numerical fractions that can be improper/topheavy. There are also **improper algebraic fractions**. An improper algebraic fraction is a fraction in which where the biggest power of x in the numerator is greater than or equal to the biggest power of x in the denominator.

Examples of improper algebraic fractions.	$\frac{x^2 + 1}{x}$,	$\frac{x^5}{x^4 + 2x + 1}$,	$\frac{x^3 + 2x + 1}{x^3 - 5x + 4}$
Examples of algebraic fractions that are <u>not</u> improper.	$\frac{x}{x^2 + 1}$,	$\frac{x^4 + 2x + 1}{x^5}$	

When we have an improper fraction, we need to be able to split it up into a quotient (something that is not a fraction), plus a proper fraction. A numerical example might be to split $\frac{23}{5}$ up to be $4 + \frac{3}{5}$. To do this with an algebraic fraction, we must use algebraic long division.

In short, the method involves repeatedly following a sequence of steps:

- (a) **divide** the first term from the numerator by the first term from the denominator.
- (b) **multiply** the result of step (a) by the full denominator.
- (c) **subtract** the result of step (b) from the numerator to create a new numerator which will be used in the next step.
- (d) Repeat steps (a), (b) and (c) until the highest power in the numerator is less than the highest power in the denominator.
 - When we stop, the final 'new' numerator is the remainder.

Example 1

Express $\frac{3x^2 + 5x + 7}{x + 1}$ as the sum of a polynomial function and a proper rational function.

Solution

Initial step: Lay out the division as shown on the right.

$$x + 1 \overline{) 3x^2 + 5x + 7}$$

Iteration 1: we follow steps a, b and c as listed above.

(continued on the next page)

(Example 1 continued)

Step 1(a): Divide the first term from the numerator ($3x^2$) by the first term from the denominator (x).

Answer: $3x$.

Write this at the top above the $3x^2$.

Step 1(b): multiply the result of step 1a ($3x$) by the full denominator ($x+1$).

Answer: $3x^2 + 3x$.

Write this below the $3x^2 + 5x + 7$

Step 1(c): subtract the result of step 1b ($3x^2 + 3x$) from the numerator ($3x^2 + 5x + 7$) to create a new numerator which will be used in the next step.

Answer: the new numerator is $2x + 7$.

Write this as a subtraction sum below the $3x^2 + 3x$.

$$\begin{array}{r} 3x \\ x+1 \overline{) 3x^2 + 5x + 7} \\ \underline{3x^2 + 3x + 0} \\ 2x + 7 \end{array}$$

The highest power in the numerator is not yet lower than the highest power in the denominator, so we continue for another iteration.

Iteration 2: we again follow steps a, b and c as listed above.

Step 2(a): Divide the first term from the new numerator ($2x$) by the first term from the denominator (x).

Answer: $(+)2$.

Write this at the very top above the $2x$

Step 2(b): multiply the result of step 1a (2) by the full denominator ($x+1$).

Answer: $2x + 2$.

Write this below the $2x + 7$.

Step 2(c): subtract the result of step 1b ($2x + 2$) from the new numerator ($2x + 7$) to create a new numerator which will be used in the next step.

Answer: the new numerator is 5 .

Write this as a subtraction sum below the $2x + 7$.

$$\begin{array}{r} 3x + 2 \\ x+1 \overline{) 3x^2 + 5x + 7} \\ \underline{3x^2 + 3x + 0} \\ 2x + 7 \\ \underline{2x + 2} \\ 5 \end{array}$$

The highest power in the numerator is now lower than the highest power in the denominator, so we stop.

When writing our answer:

- The quotient (the very top line of the written division) is $3x + 2$.
- The remainder (the final numerator) is 5 . This goes on top of the fraction part of the answer.
- The denominator of the fraction part of the answer is the denominator of the original question.

Final answer: $\frac{3x^2 + 5x + 7}{x + 1} = 3x + 2 + \frac{5}{x + 1}$.

Example 2

Express $\frac{x^3 + 4x^2 - x + 2}{x^2 + x}$ as the sum of a polynomial and a proper rational function.

Solution

The full working (following the steps outlined in the previous example) is as shown below:

$$\begin{array}{r} x + 3 \\ x^2 + x \overline{) x^3 + 4x^2 - x + 2} \\ \underline{x^3 + x^2 + 0x + 0} \\ 3x^2 - x + 2 \\ \underline{3x^2 + 3x + 0} \\ -4x + 2 \end{array}$$

Final answer: $\frac{x^3 + 4x^2 - x + 2}{x^2 + x} = x + 3 + \frac{-4x + 2}{x^2 + x}$.

Partial Fractions

You learnt at National 5 to add or subtract algebraic fractions to express them as a single fraction (possibly using a method known as 'kiss and smile'). e.g. $\frac{2}{x+3} + \frac{3}{x+4} = \frac{5x+17}{(x+3)(x+4)}$.

This method takes two (or more) 'simpler' fractions and combines them into one more complicated rational function.

The method of partial fractions goes backwards with this method: it takes a more complicated rational function and breaks it down again to be a sum of two 'simpler' fractions. We need this method in order to differentiate or integrate many functions.

The method is:

- **Initial step:** If the rational function is topheavy, use algebraic long division to re-express it.
- **Step one:** Factorise the denominator, if not already done.
- **Step two:** Identify which 'type' of factors are in the denominator (see the three types listed in this chapter).
- **Step three:** Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values A , B , C etc. in the numerators.
- **Step four:** Multiply both sides by the denominator of the original fraction.
- **Step five:** Solve equations to obtain values for A , B , C etc. The simplest way to obtain the values is to substitute in various values of x , usually the roots of the denominator. However any value of x could be substituted.

TYPE ONE: Distinct linear factors: When every factor in the denominator is linear and distinct, each factor forms the denominator of a fraction in the final answer, with a constant on top.

$$\frac{f(x)}{(x+p)(x+q)} = \frac{A}{x+p} + \frac{B}{x+q}$$

$$\frac{g(x)}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$$

For example: $\frac{x+7}{(3x-2)(x+1)} = \frac{A}{3x-2} + \frac{B}{x+1}$ or $\frac{3x+2}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$.

Example 1

Express $\frac{x+7}{x^2-x-2}$ in partial fractions.

Solution

Initial step: If the rational function is topheavy, use algebraic long division to re-express it.

This fraction is not topheavy, so this step is not required.

Step one: Factorise the denominator, if not already done.

$$\frac{x+7}{x^2-x-2} = \frac{x+7}{(x-2)(x+1)}$$

Step two: Identify which 'type' of factors are in the denominator.

Both factors are linear and distinct, so this is 'type one'.

Step three: Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values A, B, C etc. in the numerators.

$$\frac{x+7}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Step four: Multiply both sides by the denominator of the original fraction and simplify.

$$\frac{(x+7)\cancel{(x-2)(x+1)}}{\cancel{(x-2)(x+1)}} = \frac{A\cancel{(x-2)(x+1)}}{\cancel{x-2}} + \frac{B\cancel{(x-2)(x+1)}}{\cancel{x+1}}$$

$$x+7 = A(x+1) + B(x-2) \quad (**)$$

Step five: Solve equations to obtain values for A, B, C etc. The simplest way to obtain the values is to substitute in various values of x, usually the roots of the denominator. However any value of x could be substituted.

The roots of the two brackets are $x = 2$ and $x = -1$.

*First substitute $x = 2$ into the expression marked (**) above:*

$$2+7 = A(2+1) + B(2-2)$$

$$9 = A(3) + B(0)$$

$$3A = 9$$

$$A = 3$$

*Now substitute $x = -1$ into the expression marked (**) above:*

$$-1+7 = A(-1+1) + B(-1-2)$$

$$6 = A(0) + B(-3)$$

$$-3B = 6$$

$$B = -2$$

Answer: $A = 3$ and $B = -2$, so $\frac{x+7}{(x-2)(x+1)} = \frac{3}{x-2} - \frac{2}{x+1}$.

TYPE TWO: Repeated linear factors: When all factors in the denominator are linear but one of the factors appears more than once (i.e. brackets with powers, including a factor of x^n), then the factor that is repeated requires more than one partial fraction as shown below:

$$\frac{f(x)}{x^2(x+p)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+p} \qquad \frac{g(x)}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$

For example: $\frac{x^2 - 7x + 9}{(x+2)(3x-1)^2} = \frac{A}{x+2} + \frac{B}{3x-1} + \frac{C}{(3x-1)^2}$ or $\frac{2}{x^2(2x-5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x-5}$.

Example 2

Express $\frac{x^2 - 7x + 9}{(x+2)(x-1)^2}$ in partial fractions.

Solution

Initial step: If the rational function is topheavy, use algebraic long division.

This fraction is not topheavy, so this step is not required.

Step one: Factorise the denominator, if not already done.

On this occasion, the denominator is already factorised.

Step two: Identify which 'type' of factors are in the denominator.

Both factors are linear but the factor $(x-1)^2$ is repeated, so this is 'type two'.

Step three: Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values A, B, C etc. in the numerators.

$$\frac{x^2 - 7x + 9}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Step four: Multiply both sides by the denominator of the original fraction and simplify.

$$\begin{aligned} \frac{(x^2 - 7x + 9)(x+2)(x-1)^2}{(x+2)(x-1)^2} &= \frac{A(x+2)(x-1)^2}{x+2} + \frac{B(x+2)(x-1)}{x-1} + \frac{C(x+2)(x-1)^2}{(x-1)^2} \\ x^2 - 7x + 9 &= A(x-1)^2 + B(x+2)(x-1) + C(x+2) \quad (**) \end{aligned}$$

Step five: Solve equations to obtain values for A, B, C etc. The simplest way to obtain the values is to substitute in various values of x, usually the roots of the denominator. However any value of x could be substituted.

The roots of the two brackets are $x = -2$ and $x = 1$. We will also need a third value of x as there are three constants to find. We could use any value, but it makes sense to choose $x = 0$ as that is an 'easy' number.

*First substitute $x = -2$ into the expression marked (**) above:*

$$\begin{aligned} x^2 - 7x + 9 &= A(x-1)^2 + B(x+2)(x-1) + C(x+2) \quad (**) \\ (-2)^2 - 7(-2) + 9 &= A(-2-1)^2 + B(-2+2)(-2-1) + C(-2+2) \\ 4 + 14 + 9 &= A(-3)^2 + B(0)(-3) + C(0) \\ 27 &= 9A \\ A &= 3 \end{aligned}$$

(continued on next page)

*(Example 2 continued)*Now substitute $x = 1$ into the expression marked (**) above:

$$x^2 - 7x + 9 = A(x-1)^2 + B(x+2)(x-1) + C(x+2) \quad (**)$$

$$1^2 - 7(1) + 9 = A(1-1)^2 + B(1+2)(1-1) + C(1+2)$$

$$1 - 7 + 9 = A(0)^2 + B(3)(0) + C(3)$$

$$3 = 3C$$

$$C = 1$$

Now substitute $x = 0$ into the expression marked (**) above, remembering that we already know that $A = 3$ and $C = 1$:

$$x^2 - 7x + 9 = A(x-1)^2 + B(x+2)(x-1) + C(x+2) \quad (**)$$

$$0^2 - 7(0) + 9 = 3(0-1)^2 + B(0+2)(0-1) + 1(0+2)$$

$$9 = 3(-1)^2 + B(2)(-1) + 1(2)$$

$$9 = 3 - 2B + 2$$

$$9 = 5 - 2B$$

$$2B = 5 - 9$$

$$2B = -4$$

$$B = -2$$

Answer: $A = 3$, $B = -2$ and $C = 1$, so $\frac{x^2 - 7x + 9}{(x+2)(x-1)^2} = \frac{3}{x+2} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$.

An **irreducible factor** is one that cannot be factorised. A quadratic factor is irreducible if the discriminant is not a perfect square.

TYPE THREE: Irreducible quadratic factors: When one of the factors in the denominator is a quadratic that cannot be factorised (e.g. $x^2 + 5$ or $x^2 + x + 1$), then the partial fraction corresponding to that factor has a numerator of the form $Ax + B$ (as opposed to just 'A' or 'B').

For example: $\frac{3x-1}{(x+5)(x^2+3)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+3}$ or $\frac{x^2-7x+9}{(x^2+2x+8)(x-4)} = \frac{Ax+B}{x^2+2x+8} + \frac{C}{x-4}$.

Example 3Express $\frac{3x^2 + 2x + 1}{(x+1)(x^2 + 2x + 2)}$ in partial fractions.**Solution**Initial step: If the rational function is topheavy, use algebraic long division.*This fraction is not topheavy, so this step is not required.*Step one: Factorise the denominator, if not already done.*On this occasion, the denominator is already factorised.*Step two: Identify which 'type' of factors are in the denominator.*(continued on next page)*

(Example 3 continued)

There is a quadratic factor $x^2 + 2x + 2$. We double-check that this is irreducible using the discriminant:

$$b^2 - 4ac = 2^2 - 4 \times 1 \times 2 = -4$$

$$b^2 - 4ac < 0, \quad \text{so } x^2 + 2x + 2 \text{ is irreducible, so this is 'type three'.$$

Step three: Based on which 'type' of factors are involved, write the form that the final answer will take, with unknown values A , B , C etc. in the numerators.

$$\frac{3x^2 + 2x + 1}{(x+1)(x^2 + 2x + 2)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 2x + 2}$$

Step four: Multiply both sides by the denominator of the original fraction and simplify.

$$\frac{(3x^2 + 2x + 1)(x+1)(x^2 + 2x + 2)}{(x+1)(x^2 + 2x + 2)} = \frac{A(x+1)(x^2 + 2x + 2)}{x+1} + \frac{(Bx + C)(x+1)(x^2 + 2x + 2)}{x^2 + 2x + 2}$$

$$3x^2 + 2x + 1 = A(x^2 + 2x + 2) + (Bx + C)(x + 1)$$

Step five: Solve equations to obtain values for A , B , C etc.

The root of the linear bracket is $x = -1$. The irreducible factor has no roots, so we need two further x -values to substitute. We could use any value, but it makes sense to choose $x = 0$ and $x = 1$ as they are 'easy' numbers.

First substitute $x = -1$ into the expression marked (**) above:

$$3x^2 + 2x + 1 = A(x^2 + 2x + 2) + (Bx + C)(x + 1)$$

$$3(-1)^2 + 2(-1) + 1 = A((-1)^2 + 2(-1) + 2) + (B(-1) + C)(-1 + 1)$$

$$3 - 2 + 1 = A(1 - 2 + 2) + (-B + C)(0)$$

$$2 = A$$

Now substitute $x = 0$ into the expression marked (**) above remembering that we already know that $A = 2$:

$$3x^2 + 2x + 1 = A(x^2 + 2x + 2) + (Bx + C)(x + 1)$$

$$3(0)^2 + 2(0) + 1 = 2(0^2 + 2(0) + 2) + (B(0) + C)(0 + 1)$$

$$1 = 2(2) + (C)(1)$$

$$1 = 4 + C$$

$$C = -3$$

Now substitute $x = 1$ into the expression marked (**) above, remembering that we already know that $A = 2$ and $C = -3$:

$$3x^2 + 2x + 1 = A(x^2 + 2x + 2) + (Bx + C)(x + 1)$$

$$3(1)^2 + 2(1) + 1 = 2(1^2 + 2(1) + 2) + (B(1) - 3)(1 + 1)$$

$$3 + 2 + 1 = 2(5) + (B - 3)(2)$$

$$6 = 10 + 2(B - 3)$$

$$B = 1 \quad (\text{some working omitted})$$

Answer: $A = 2$, $B = 1$ and $C = -3$, so $\frac{3x^2 + 2x + 1}{(x+1)(x^2 + 2x + 2)} = \frac{2}{x+1} + \frac{x-3}{x^2 + 2x + 2}$.

When the fraction is improper, algebraic division (see page 11) must be used first to express the improper fraction as a polynomial and a proper fraction.

Example 4

Express $\frac{x^3 - 3x}{x^2 - x - 2}$ as the sum of a polynomial and partial fractions.

Solution

Step A: use algebraic long division:

$$\begin{array}{r} x + 1 \\ x^2 - x - 2 \overline{) x^3 + 0x^2 - 3x + 0} \\ \underline{x^3 - x^2 - 2x + 0} \\ x^2 - x + 0 \\ \underline{x^2 - x - 2} \\ 2 \end{array}$$

Hence $\frac{x^3 - 3x}{x^2 - x - 2} = x + 1 + \frac{2}{x^2 - x - 2}$. (*)

Step B: Express $\frac{2}{x^2 - x - 2}$ in partial fractions.

Using the standard method from page 14 (this is a 'type one' fraction).

$$\frac{2}{x^2 - x - 2} = \frac{2}{(x-2)(x+1)}$$

$$\frac{2}{x^2 - x - 2} = \frac{2}{(x-2)(x+1)}$$

$$\frac{2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$2 = A(x+1) + B(x-2) \quad (\text{a step of working ('cancelling') was omitted here})$$

When $x = 2$:

$$2 = A(2+1) + B(2-2)$$

$$2 = 3A + 0$$

$$A = \frac{2}{3}$$

when $x = -1$:

$$2 = A(-1+1) + B(-1-2)$$

$$2 = 0 - 3B$$

$$B = -\frac{2}{3}$$

Hence $\frac{2}{x^2 - x - 2} = \frac{\frac{2}{3}}{x-2} - \frac{\frac{2}{3}}{x+1} = \frac{2}{3(x-2)} - \frac{2}{3(x+1)}$.

Step C: go back to formula (*) to re-express the full function:

Final answer: $\frac{x^3 - 3x}{x^2 - x - 2} = x + 1 + \frac{2}{3(x-2)} - \frac{2}{3(x+1)}$.