Example 1

Integrate $\displaystyle\int \frac{5}{x^2+49}\,dx$

Solution

$$\begin{aligned}\int \frac{5}{x^2+49}\,dx &= \int \frac{5}{7^2+x^2}\,dx \\ &= 5\cdot\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right)+C \\ &= \underline{\underline{\frac{5}{7}\tan^{-1}\left(\frac{x}{7}\right)+C}}\end{aligned}$$

Example 2

Integrate $\displaystyle\int \frac{6}{\sqrt{4-9x^2}}\,dx$

Solution

$$\int \frac{6}{\sqrt{4-9x^2}}\,dx = \int \frac{6}{\sqrt{2^2-(3x)^2}}\,dx$$

Using the substitution $u = 3x$ (using the technique of integration by substitution outlined on page 34) we have $dx = \frac{du}{3}$ and then:

$$\begin{aligned}\int \frac{6}{\sqrt{2^2-(3x)^2}}\,dx &= \int \frac{6}{\sqrt{2^2-u^2}}\cdot\frac{du}{3} \\ &= \frac{6}{3}\int \frac{1}{\sqrt{2^2-u^2}}\,du \\ &= 2\cdot\sin^{-1}\left(\frac{u}{2}\right)+C \qquad \text{using } \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C \\ &= \underline{\underline{2\sin^{-1}\left(\frac{3x}{2}\right)+C}} \qquad \text{resubstituting } u=3x\end{aligned}$$

Integrating Rational Functions

Some rational functions are standard integrals that we can integrate on sight:

- $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$
- $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right|+C$

Other more complicated rational functions that are not in this form must be split up before we can integrate. There are two ways of splitting up:

- If the denominator <u>can</u> be factorised, use partial fractions (see page 13).
- If the denominator <u>cannot</u> be factorised, split up the numerator so that each term in the numerator becomes the numerator of 'its own' fraction, e.g. $\frac{x+3}{x^2+4} = \frac{x}{x^2+4} + \frac{3}{x^2+4}$.

