Advanced Higher Mathematics: Formulae

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Green (G): Formulae you must memorise in order to pass Advanced Higher maths as they are not on the formula sheet.
Amber (A): These formulae are given on the formula sheet. But it will still be useful for you to memorise them.
Red (R): Don't worry about memorising these, but they might be useful to save time in classwork and homework.

Trigonometric Identities: (including those from National 5 and Higher)

	Essential Formulae to know off by heart for the exam (G)	Other useful ones that may be useful for homework/classwork etc.
Links between ratios	$\cos^2 A + \sin^2 A = 1$ $\tan A = \dfrac{\sin A}{\cos A}$	$1 + \tan^2 A = \sec^2 A$ $\cot^2 A + 1 = \operatorname{cosec}^2 A$
Squared	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	
Compound Angle	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
Double Angle	$\sin(2A) = 2\sin A \cos A$ $\cos(2A) = \cos^2 A - \sin^2 A$	$\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}$

Exact Values (you should know all these for the non-calculator paper)

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.	0	undef.	0

Negative facts:
$\sin(-\vartheta) = -\sin(\vartheta)$
$\cos(-\vartheta) = +\cos(\vartheta)$
$\tan(-\vartheta) = -\tan(\vartheta)$

Complex Numbers

For the complex number, $z = a + bi$,

- the **modulus** is given by $|z| = \sqrt{a^2 + b^2}$
- and the **argument** is given by $\tan\vartheta = \dfrac{b}{a}$ $\quad -\pi < \vartheta < \pi$
- The **conjugate** is $\bar{z} = a - bi$

De Moivre's Theorem says that

for any $z = r(\cos\vartheta + i\sin\vartheta)$, then $z^n = r^n(\cos n\vartheta + i\sin n\vartheta)$ $\;(n \in \mathbb{Q})$

Differentiation

Product Rule: $u\frac{dv}{dx}+v\frac{du}{dx}$ **Quotient Rule:** $\dfrac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$f(x)$	$f'(x)$
$\sin^{-1}x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\dfrac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\ln x \quad x>0$	$\dfrac{1}{x}$
e^x	e^x

$f(x)$	$f'(x)$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$

To differentiate an inverse function: $\dfrac{dx}{dy}=\dfrac{1}{\frac{dy}{dx}}$

Parametric Equations (where $x=f(t), y=g(t)$):

- Gradient (direction of movement) = $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- Speed = $\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}$
- $\dfrac{d^2y}{dx^2}=\dfrac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3}$

Integration

On Formula Sheet

$f(x)$	$\int f(x)dx$
$\sec^2 ax$	$\frac{1}{a}\tan(ax)+C$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+C$
$\dfrac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$
e^{ax}	$\frac{1}{a}e^{ax}+C$

To save you time in harder questions for homework/classwork, <u>no need to memorise</u>:

$f(x)$	$\int f(x)dx$
$\tan x$	$\ln\lvert\sec x\rvert+C$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert+C$
$\cot x$	$\ln\lvert\sin x\rvert+C$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert+C$

Integration by Parts

$$\int u\frac{dv}{dx}\,dx = uv-\int v\frac{du}{dx}\,dx$$

Volume of **solid of revolution** *f(x)* between *a* and *b*:

About *x* axis: $V=\pi\int_a^b f(x)^2dx$ About *y* axis: $V=\pi\int_a^b f(y)^2dy$

Sequences and Series

	Arithmetic Series	Geometric Series
n^{th} term	$u_n = a + (n-1)d$	$u_n = ar^{n-1}$
Sum of n terms	$S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$	$S_n = \frac{a(1-r^n)}{1-r}$ $\quad r \neq 1$
Sum to infinity	n/a	$S_\infty = \frac{a}{1-r}$ $\quad \lvert r \rvert < 1$

Important Identities

$$\sum_{k=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2} \qquad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

and in particular:

<u>Very</u> useful to memorise:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Less essential to memorise:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Functions

Odd function: $f(-x) = -f(x)$ **Even function:** $f(-x) = f(x)$

(180° rotational symmetry) (line symmetry about the y-axis)

Binomial Theorem

The coefficient of the r^{th} term in the binomial expansion $(a+b)^n$ is $\binom{n}{r} a^{n-r} b^r$

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Vectors, Lines and Planes

Angle between two vectors: (Higher) $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\vartheta$

Equations of a 3d line: through (x_1, y_1, z_1) and with direction vector $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Parametric form

$x = x_1 + at$

$y = y_1 + bt \quad (\mathbf{x} = \mathbf{a} + t\mathbf{d})$

$z = z_1 + ct$

Symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} (= t)$$

Equations of a plane:

Normal **n** is $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

Point on line = P (with position vector **a**)

Vector equation

$\mathbf{x} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$

Symmetric/Cartesian

$lx + my + nz = k$

where $k = \mathbf{a} \bullet \mathbf{n}$

Parametric (A)

$\mathbf{x} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$

(**b** and **c** are any two non-parallel vectors in plane)

Angle between two lines = Acute angle between their direction vectors.

Angle between two planes = Acute angle between their normal.

Angle between line and plane = 90° – (Acute angle between **n** and **d**)

Cross (vector) product:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\vartheta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Scalar triple product:

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Matrices

		Determinant and Inverse
2×2 matrices	$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\det A = ad-bc$ and $A^{-1}=\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
3×3 matrices	$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$(AB)^{-1}=B^{-1}A^{-1}$ $\quad$ $(AB)^T = B^T A^T$ $\quad$ $\det AB = \det A \det B$ (A)

Transformation Matrices

Anti-CW Rotation by ϑ degrees $\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$,

Reflection in y-axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Dilatation by scale factor a $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$,

Reflection in x-axis $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Differential Equations

For $\dfrac{dy}{dx}+P(x)y=Q(x)$, the Integrating Factor $I(x)$ is $e^{\int P(x)dx}$

and the solution is given by $I(x)y=\int I(x)Q(x)dx$

Second Order Differential Equations

COMPLEMENTARY FUNCTION (Homogeneous Equations)

Nature of roots	Form of general solution
Two distinct real m and n	$y = Ae^{mx} + Be^{nx}$
Real and equal m	$y=(A+Bx)e^{mx}$
Complex conjugate $m = p \pm iq$	$y = e^{px}(A\cos qx + B\sin qx)$

PARTICULAR INTEGRAL (Inhomogeneous Equations)

Right-hand side contains…	For Particular Integral, try…
sin ax or cos ax	$y = P\cos ax + Q\sin ax$
e^{ax}	$y = Pe^{ax}$
Linear expression $y = ax+b$	$y = Px+Q$
Quadratic expression $y = ax^2+bx+c$	$y = Px^2+Qx+R$